Tanzania Journal of Science 47(2): 728-737, 2021
ISSN 0856-1761, e-ISSN 2507-7961

Mathematical Analysis of Harvested Predator-Prey System with Prey Refuge and Intraspecific Competition

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Received 13 Jan 2021, Revised 12 May 2021, Accepted 19 May 2021, Published May 2021
DOI: https://dx.doi.org/10.4314/tjs.v47i2.28

Abstract

In this paper, a predator-prey relationship in the presence of prey refuge was studied. The analysis of the dependence of locally stable equilibrium points on the parameters of the problem was carried out. Bifurcation and limit cycles for the model were analyzed to show the dynamical behaviour of the system. The results showed that the system is stable at a constant prey refuge $m = 0.3$ and prey harvesting rate $H = 0.3$. However, increasing m and decreasing H or vice versa, the predator-prey system remains stable. It was further observed that for a constant prey refuge $m \geq 0.78$, the predator population undergoes extinction. Therefore, m was found to be a bifurcation parameter and $m = 0.78$ is a bifurcation value.

Keywords: Prey refuge, bifurcation, harvesting, intraspecific competition, phase portrait.

Introduction

The dynamic relationship between predators and their preys has been, and continue to be, one of the most important topics of study in ecology and mathematical ecology due to its universal existence and importance (Berryman 1992, Mapunda et al 2018). As species interact, they compete amongst themselves for resources and space. They are doing so either under intraspecific competition or interspecific competition (Schwinning and Kelly 2013). A Holling type I functional response assumes that there is no satiation (linear relationship) rather predators keep on feeding until the prey population goes to extinction or remains with few individuals to make preying difficult (Dawes and Souza 2013). This situation leads to competition for resources among the predators. A prey refuge is one of the strategies that can be used to protect preys from over-predation (Sagamiko et al. 2015). Increasing prey refuge implies increasing prey density (Kar 2005).

Bifurcation is a sudden change of stability behaviour of a dynamical system at equilibrium as an equilibrium parameter changes. For complex dynamical systems, bifurcation theory and stability analysis are used for investigating their dynamical behaviour without determining explicitly the solutions of their governing equations for various initial and boundary conditions (Stefanou and Alevizos 2016).

Much work has been done on predator-prey systems incorporating prey refuge and species competition as pointed out by Majeed (2018), Kar (2005), Kar (2006), Das et al. (2013) and

Sagamiko et al. (2015). Majeed (2018), studied the dynamics of a prey-predator model with refuge and stage structures in both populations, Sagamiko et al. (2015) considered a threatened prey-predator system with prey refuge in the Serengeti ecosystem, Kar (2005) worked on the stability analysis of a prey-predator model incorporating a prey refuge, Kar and Chakraborty (2010) explored the effort dynamics in a prey-predator model with harvesting in which competition among predators was considered. In particular, Kar (2006) and Das et al. (2013) investigated the dynamics of exploited prey-predator with constant prey refuge. However, the study of the dynamical behaviour of a harvested Holling type I predator-prey system with prey refuge incorporating intraspecific competition among predator species is not treated in the existing literature, and therefore this study intended to take care of this aspect.

This paper is organized as follows: Model formulation and basic results are under the Materials and Methods section where the model is formulated and the existence of equilibrium points is studied, Boundedness of the system, stability analysis of the equilibrium points, and existence of periodic solutions are presented. The next section is about numerical solutions in which phase plots and bifurcation diagrams for examining the effects of harvesting and prey refuge to the dynamics of the prey-predator system are presented followed by discussion and concluding remarks of the work.

Materials and Methods

The mathematical model

A predator-prey system is modelled using Lotka-Voltera as follows:

$$\frac{dN}{dt} = rN\left(1-\frac{N}{K}\right) - \beta_1(1-m)NP - HN \qquad (1)$$
$$\frac{dP}{dt} = \beta_2(1-m)NP - \mu P - \alpha P^2$$

where N and P denote prey and predator population densities, respectively at any time t, and r, K, β_1, β_2, μ, α, and H are all positive constants. Here r represents intrinsic growth rates of prey population; K is the carrying capacity; β_1 is the predation rate to prey species; μ is the natural mortality rate for a predator; β_2 is the predator biomass to the prey species; α is the intraspecific competition among predators; H is harvesting rate in prey and m is prey refuge constant with $m \in [0,1]$. The following assumptions have been taken into considerations of the model system (1): There is logistic growth of prey species in absence of predation and external factors. The rate of increase of the predator population depends on the amount of prey biomass it converts to food.

Existence of equilibrium points

To study the existence of the equilibrium points of the system (1), all possible solutions satisfying equation (2) are listed.

$$rN\left(1-\frac{N}{K}\right) - \beta_1(1-m)NP - HN = 0 \qquad (2)$$
$$\beta_2(1-m)NP - \mu P - \alpha P^2 = 0$$

There is a trivial equilibrium point $E_0(0,0)$. The next two non-trivial equilibrium points are $E_1 = \left(K - \frac{KH}{r}, 0\right)$ which exists if $r > H$, that is if the harvesting rate to prey population does not exceed its intrinsic growth rate, and the interior point

$E_2\left(\frac{\alpha K(r-H) - (\mu)(Km\beta_1 - K\beta_1)}{\alpha r + (Km\beta_1 - K\beta_1)(m\beta_2 - \beta_2)}, \frac{K\beta_2(r-H)(1-m) - r(\mu)}{\alpha r + K\beta_1\beta_2(1-m)^2}\right)$ which exists if $r > H$ and $0 \le m < 1 - \frac{r(\mu)}{K\beta_2(r-H)}$.

Boundedness of solution

The state variables of model (1) represent the population densities of species. Thus, they must be non-negative at any time $t \geq 0$.

Lemma 1. All solutions of the system (1) are bounded in $\Re_+^2 = \{(N,P): N > 0, P > 0\}$.

Proof: Defining a function represented in Equation (3):

$$S(N,P) = N + \frac{\beta_1}{\beta_2} P \qquad (3)$$

Differentiating S along the trajectory; resulting into Equation (4) as:

$$S' = N' + \frac{\beta_1}{\beta_2} P' \qquad (4)$$

Using Equation (1) in (4) results to

$$S'(t) = (r - H)N - \frac{rN^2}{K} - \frac{\beta_1}{\beta_2}\left[\alpha P^2 + (\mu)P\right] \qquad (5)$$

Taking the sum of the last relation and the Equation (3) multiplied by ν results into;

$$S' + \nu S = (r - H + \nu)N - \frac{rN^2}{K} - \frac{\beta_1}{\beta_2}\left[\alpha P^2 + (\mu - \nu)P\right]$$

$$\leq \frac{K}{4r}(r - H + \nu)^2 + \frac{\beta_2}{4\alpha\beta_1}[(\mu + \nu)]^2$$

$$\leq M \qquad (6)$$

Solving for S as $t \to 0$, obtaining Equation (7) as:

$$0 \leq S(N,P) \leq \frac{M}{\nu} + \left(S(0) - \frac{M}{\nu}\right)e^{-\nu t} \qquad (7)$$

as $t \to \infty$ gives

$$0 \leq S(N,P) \leq \frac{M}{\nu} \qquad (8)$$

Thus, S is bounded and this completes the proof.

Linear stability analysis

To study the linear stability of the equilibrium points, the Jacobian matrix of the right-hand side of Equation (1) is presented in Equation (9).

$$J(E_i) = \begin{bmatrix} r\left(1 - \frac{N}{K}\right) - \frac{rN}{K} - \beta(1-m)P - H & -\beta_1(1-m)N \\ \beta_2(1-m)P & \beta_2(1-m)N - \mu - 2\alpha P \end{bmatrix} \qquad (9)$$

It can be shown that the equilibrium points E_0 and E_1 are always unstable.

The equilibrium point E_0 has eigenvalues $(r - H, -\mu)$, and therefore unstable for $r > H$. The equilibrium point E_1 has eigenvalues $\left(H_1 - r, K\beta_2(1-m)\left(1 - \frac{H}{r}\right) - \mu\right)$ and unstable for $r > H$ and $K\beta_2(1-m)\left(1 - \frac{H}{r}\right) > \mu$.

The Jacobian matrix of the interior point E_2 takes the form;

$$J(E_3) = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \qquad (10)$$

The characteristics polynomial of the matrix in Equation (10) is

$$\lambda^2 - (A_{11} + A_{22})\lambda + (A_{11}A_{22} - A_{12}A_{21}) = 0 \quad (11)$$

Where;

$$A_{11} = -\left[\frac{r[\alpha r - \alpha H - (m-1)(\mu)\beta_1][\alpha r + K(m-1)^2\beta_1\beta_2]}{[\alpha r + (m-1)^2\beta_1\beta_2]^2}\right],$$

$$A_{12} = -\left[\frac{K(m-1)\beta_1[\alpha H - \alpha r + (m-1)(\mu)\beta_1]}{\alpha r + K(1-m)^2\beta_1\beta_2}\right],$$

$$A_{21} = \frac{(m-1)\beta_2[r(\mu) + K(m-1)(r-H)\beta_2]}{\alpha r + K(1-m)^2\beta_1\beta_2},$$

$$A_{22} = \frac{K(1-m)\alpha(r-H)\beta_2 - \alpha r\mu}{\alpha r + K(m-1)^2\beta_1\beta_2} + \frac{2\alpha r\mu - 2K(1-m)\alpha(r-H)\beta_2}{[\alpha r + K(m-1)^2\beta_1\beta_2]^2} + B, \text{ such that}$$

$$B = \frac{[\alpha^2 r^2 + K(m-1)^2\alpha r\beta_1\beta_2]}{[\alpha r + K(m-1)^2\beta_1\beta_2]^2}.$$

Since $A_{11} < 0$ and $A_{12} < 0$ then, linear stability takes place if $A_{22} < 0$ and $A_{21} > 0$.

Existence of periodic solutions

The system of Equation (1) can be written in general form as;

$$\frac{dN}{dt} = f(N,P)$$
$$\frac{dP}{dt} = g(N,P) \quad (12)$$

Where,

$$f(N,P) = rN\left(1 - \frac{N}{K}\right) - \beta_1(1-m)NP - HN,$$

and

$$g(N,P) = \beta_2(1-m)NP - \mu P - \alpha P^2.$$

According to Bendixson's negative criterion, the following theorem holds:

Theorem:

The system (12) can have periodic solutions if $\frac{\partial f}{\partial N}(N,P) + \frac{\partial g}{\partial P}(N,P)$ changes sign or equal to zero.

By applying this theorem to the system (1), it can be shown that

$\frac{\partial f}{\partial N}(N,P) + \frac{\partial g}{\partial P}(N,P) \neq 0$ and changes sign. Therefore, the system has periodic solutions.

Numerical Results and Discussion

The simulation of the model (1) is done using parameter values as given in Table 1.

Table 1: Parameter values for model (1)

Parameter	Parameter value	Source
r	2	Kar (2004)
β_1	0.6	Kar (2003)
β_2	0.4	Assumed
m	$0 \leq m < 1$	Standard range for refuge
K	200	Assumed
H	0.24	Das et al. 2013
α	0.083	Kar and Chakraborty (2010)
μ	0.0001	Assumed

Bifurcation

Figure 1 shows the bifurcation behaviour of the system (1) with *m* as a bifurcation parameter. For $m > 0.78$, there is an increase in prey population and decrease in predator population leading to extinction due to resource scarcity. The decrease in predator population due to more protected prey is also described in Figure 8.

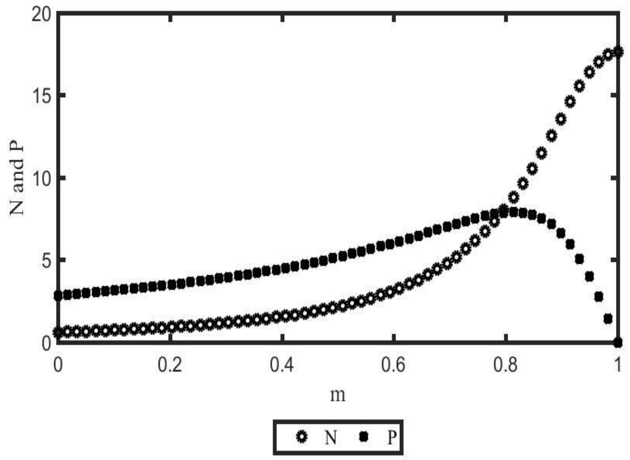

Figure 1: A bifurcation diagram for system (1) with m a bifurcation parameter. The vertical axis measures both N^* and P^*.

Phase portraits

Figure 2 shows a phase diagram that justifies the existence of periodic solutions as was done in theoretical analysis. Moreover, it is observed that the interior equilibrium point is always stable when *m* = 0.3 and *H* = 0.3; the behaviour which is also displayed in Figure 7.

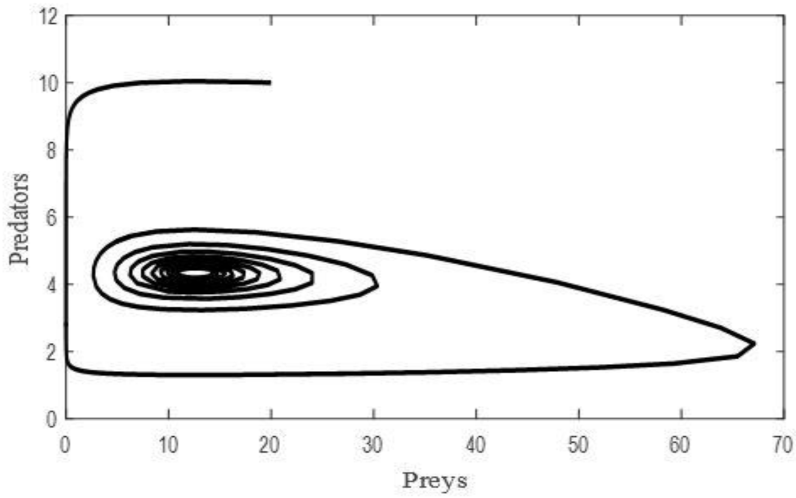

Figure 2: Phase portrait of the system (1) for m = 0.3, H = 0.3.

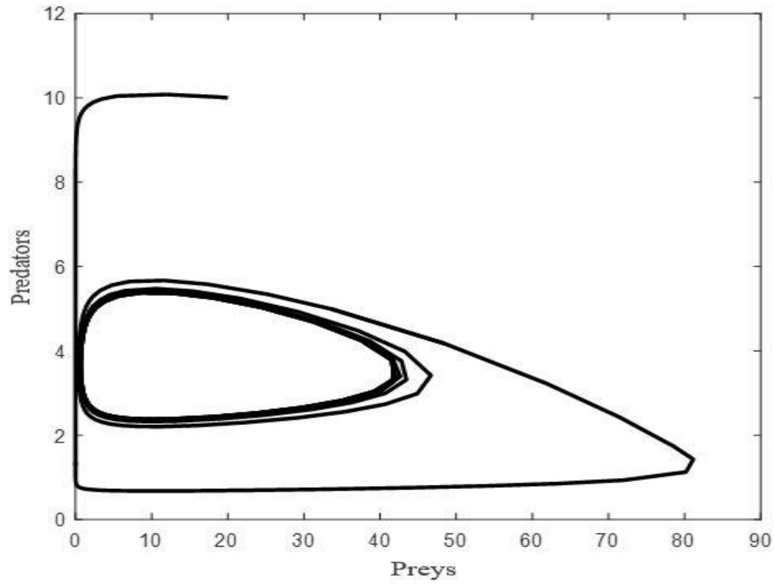

Figure 3: Phase portrait of the system (1) when m = 0.13, H = 0.1.

Figure 3 shows the effect of a decrease in refuge and harvesting. It is observed that the predator-prey system is unstable. The instability is due to excess preys resulted from small protected prey species. To bring the system into stability, the harvesting rate has to be increased as shown in Figure 4.

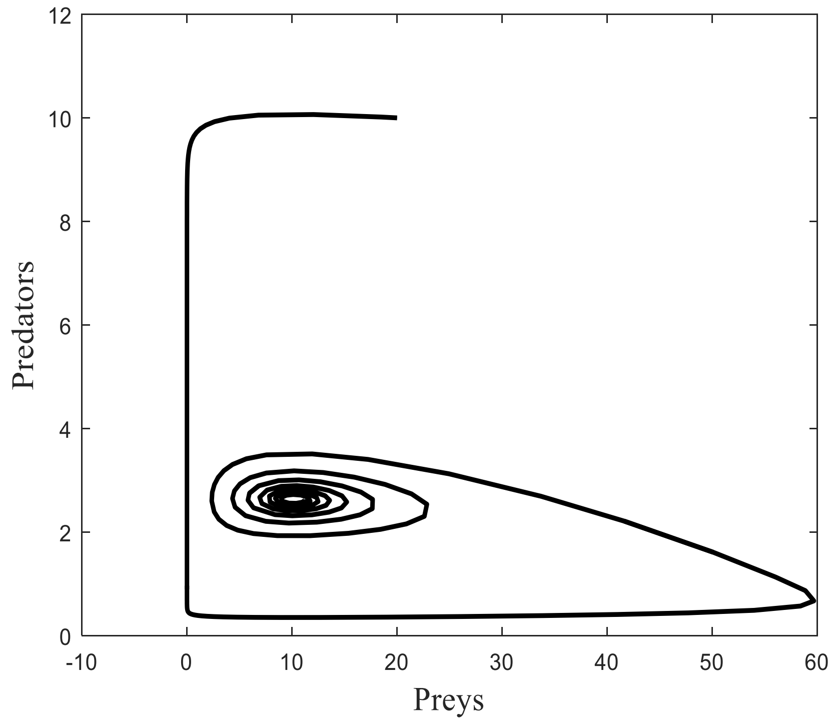

Figure 4: Phase portrait of the system (1) when m = 0.13, H = 0.63.

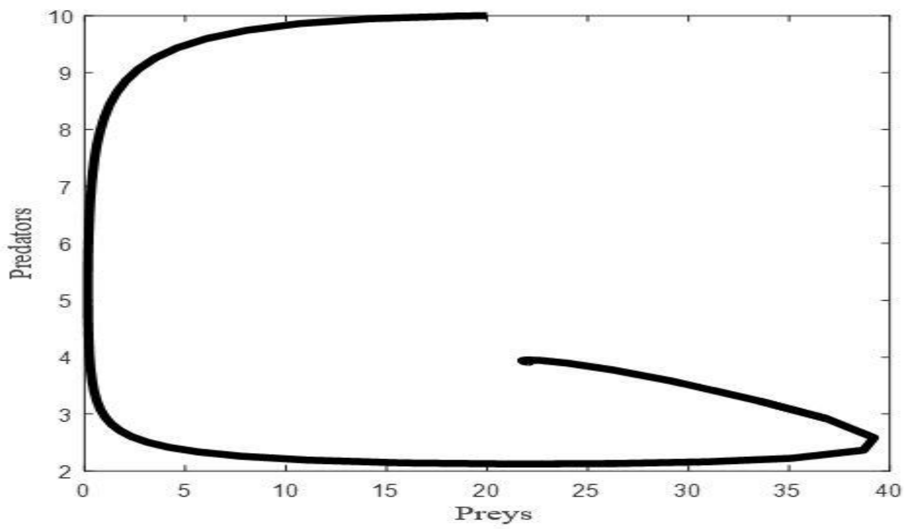

Figure 5: Phase portrait of the system (1) when m = 0.5, H = 0.63.

Figure 5 portrays the effect of increasing prey refuges for $m > 0.4$ and harvesting rate above 50% (i.e., $H = 0.63$ for this case). It reveals the instability of the system which can be corrected by decreasing the harvesting rate to unprotected preys as shown in Figure 6.

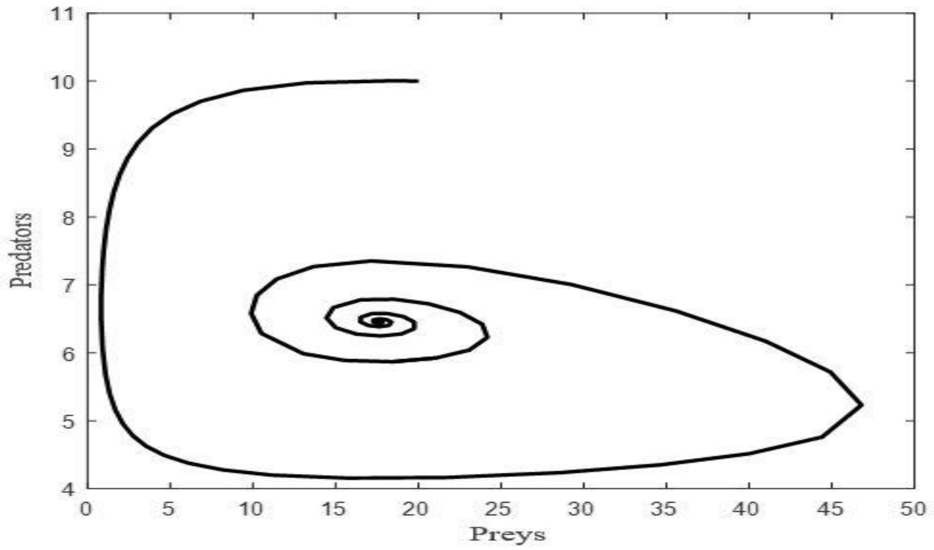

Figure 6: Phase portrait of the system (1) when m = 0.5, H = 0.01.

Species interactions

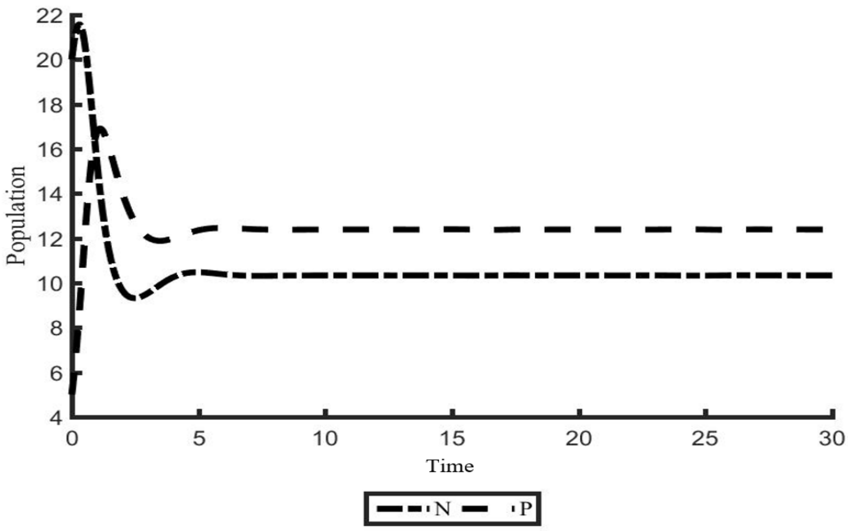

Figure 7: Predator-prey interaction for protected prey $m < 0.3$.

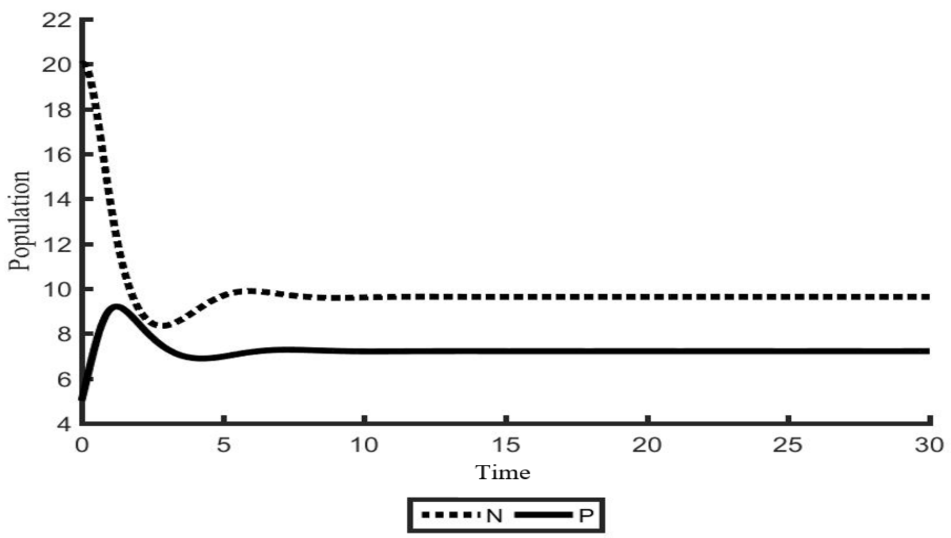

Figure 8: Predator-prey interaction for increased protected prey $m > 0.5$.

Prey harvesting

Figure 9 demonstrates the extinction of both species when the rate of harvesting exceeds prey intrinsic growth rate. This results remind of the need for abiding with the theoretical analysis that the harvesting rate should not exceed prey intrinsic growth rate for the predator-prey system to be stable.

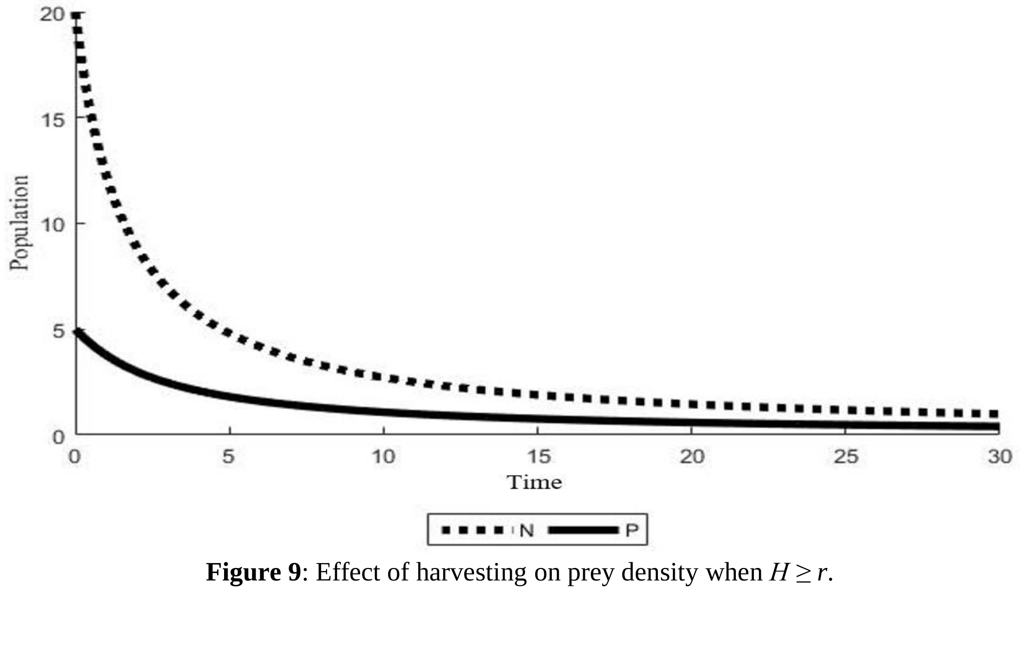

Figure 9: Effect of harvesting on prey density when $H \geq r$.

Conclusion

In this work, a dynamic behaviour of a harvested predator-prey system with prey refuge and intraspecific competition was studied. The model incorporated harvesting to prey and intraspecific competitions to predator individuals. It was also assumed that Holling type I is the predator's functional response to preys. Numerical studies have been conducted to verify the theoretical results. The dynamical behaviour indicated by simulations is in agreement with the theoretical results.

Acknowledgements

Authors acknowledge their institutions for the provision of research resources.

Declaration: The authors have no conflict of interest.

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