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Determination of Cramer-Rao Lower Bound (CRLB) and Minimum Variance Unbiased Estimator of a DC Signal in AWGN Using Laplace Transform

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Abstract

This paper presents an alternative approach for the determination of Cramer-Rao Lower Bound (CRLB) and Minimum Variance Unbiased Estimator (MVUE) using Laplace transformation. In this work, a DC signal in Additive White Gaussian Noise (AWGN) was considered. During the investigation, a number of experiments were conducted to analyze different possible outputs under different conditions, and then the patterns of the outcomes were studied. Finally closed-form expressions for the CRLB and MVUE were deduced employing the Laplace transformation. The resulting expressions showed that the proposed method has almost the same number of steps as the existing method. However, the latter requires only the knowledge of algebra to arrive at the CRLB expressions contrary to the existing approach where a strong mathematical background is required and hence making it superior over the existing method, in that sense.

Keywords: Additive White Gaussian Noise, Cramer-Rao Lower Bound, DC-Value, Laplace transform, and Minimum Variance Unbiased Estimator (MVUE).

Introduction

In communication and signal processing, it is common to encounter signal estimation problems. In so doing, it is very useful to place a lower bound on variance of any unbiased estimator in use. This lower bound can easily help select the best estimator for a given problem in hand based on the minimum variance it can attain. On the other language, the bound tells how best one can achieve during a given estimation problem (Alslaimy and Smith 2019, Huang et al. 2020, Hung et al. 2020, Kay 1993, Khorasani et al. 2020, Li et al. 2021, Mao et al. 2020, Tian et al. 2020).

There are many lower bounds in the literature, but the most famous bound, attractive and relatively easy to evaluate is the Cramer- Rao Lower Bound (CRLB). The CRLB, which is defined as the inverse Fisher Information Matrix (FIM), is a useful tool to find the smallest variance estimates for unbiased estimators. In radar applications in which the CRLB depends on the geometry of the bistatic configuration, it is normally used to determine the optimal transceiver locations so that the estimation accuracy is improved (Alslaimy and Smith 2019). Among its features, CRLB can readily give a Minimum Variance Unbiased Estimator (MVUE), if it exists (Alslaimy and Smith 2019, Kay 1993, Tian et al. 2020).

From the literature, the CRLB determination requires good knowledge of calculus to evaluate the second derivative, which is not only cumbersome, but error-prone. Furthermore, a strong knowledge of probability and statistics is necessary as well to help evaluate the expected value. It is true, with no doubt, that probability is marked by many scholars as among the difficult branches in mathematics (Kay 1993, Tian et al. 2020). In short, even though the CRLB is preferred compared to other lower bounds existing in the literature, its procedure is not straightforward for the reasons outlined earlier (Kay 1993, Khorasani et al. 2020, Mao et al. 2020).

The Laplace transform has a number of engineering applications and found considerable attention in science and engineering due to its capability to solve complex engineering problems. The Laplace transform, named after Pierre-Simon Laplace, is an integral transform that converts a function of a real variable to a function of a complex variable. The Laplace transform is a tool for solving differential equations and in particular, it transforms differential equations into algebraic equations and convolution into multiplication (Phillips and Parr 2008, Schiff 1999). Laplace transform has also found applications in probability theory. In pure and applied probability, the Laplace transform is defined as an expected value of a given random variable, say X. By convention, this is referred to as the Laplace transform of the random variable X itself. Here, replacing s by $-t$ gives the moment generating function of X (Williams 1973, Schiff 1999, Phillips and Parr 2008).

The objective of this work was therefore, to investigate the possibility of utilizing the capability of Laplace transformation to determine the CRLB of parameter estimators as well as the determination of their corresponding Minimum Variance Unbiased Estimators (MVUEs). With the application of Laplace transform, all complicated procedures including second derivative will be mapped into simpler algebraic manipulations (Schiff 1999).

Materials and Methods

Cramer-Rao lower bound

In a nutshell, CRLB states that the variance of any unbiased estimator is at least as high as the inverse of the Fisher information measure. Consider a DC signal contaminated with Additive White Gaussian Noise (AWGN) having zero mean and known variance. If N observations were made during the estimation process, the data model can be expressed mathematically as shown in Equation (1) (Kay 1993):

$$x(n) = A + w(n) \qquad (1)$$

where, $w(n) \sim G(0, \sigma^2)$, σ^2 is the known variance of the noise $w(n)$, A is the DC signal to be estimated and $n = 0,1,2 \cdots N-1$. Assuming all observations are independent and identically distributed (iid), then the CRLB for the estimated DC signal, $\hat{A}$, is given by Equation (2):

$$Var(\hat{A}) \geq \frac{1}{I(A)} \qquad (2)$$

and $$I(A) = -E\left[\frac{\partial^2 \ln p(\underline{x}, A)}{\partial A^2}\right] \qquad (3)$$

where $I(A)$ is the Fisher Information measure, and $\ln p(\boldsymbol{x}, A)$ is the log-likelihood of the pdf of $\boldsymbol{x}$ indexed by A, which is taken to be Gaussian in this study.

From the given expressions, it is clear that the CRLB determination requires a good knowledge of calculus and probability theories making it difficult to evaluate. This calls for an alternative, tractable approach, and in this paper, Laplace transform is used to address the challenges.

Proposed approach for CRLB determination

A. CRLB for $\alpha = A$

Consider a DC signal, A, contaminated with AWGN having zero mean and known variance, σ^2. If multiple observations were made during the estimation process, the data model can be expressed mathematically as Equation (1). Assuming all observations are iids. The following proposed theorems help to determine the CRLB for A:

Theorem 1

Given a DC signal, A defined under f(A) corrupted by AWGN with zero mean and known variance, σ^2. If multiple observations made are iid's then CRLB and MVUE of A are respectively.

$$CRLB_{\hat{A}} = -\frac{1}{f''(A)} = -\frac{1}{L^{-1}[s^2F(s) - sf(0) - f'(0)]} \quad (4)$$

and

$$\hat{A} = \frac{f'(0)}{f''(A)} = \frac{f'(0)}{L^{-1}[s^2F(s) - sf(0) - f'(0)]} \quad (5)$$

with

$$F(s) = -\frac{N}{2\sigma^2}\left[\frac{\bar{x}^2}{s} - \frac{2\bar{x}}{s^2} + \frac{2}{s^3}\right] \quad (6)$$

$$f(0) = -\frac{N\bar{x}^2}{2\sigma^2} \quad (7)$$

and

$$f'(0) = \frac{N\bar{x}}{\sigma^2} \quad (8)$$

Proof

Consider a DC signal, A, contaminated with Additive White Gaussian Noise (AWGN) having zero mean and known variance, σ^2. If multiple observations were made during the estimation process, the data model expressed mathematically as in Equation (1). Assuming all observations are independent and identically distributed (iid). To determine the CRLB for A Equation (9) can be used.

$$p(\boldsymbol{x}, A) = \frac{1}{(2\pi\sigma^2)^{N/2}} \exp\left[-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1}(x[n] - A)^2\right] \quad (9)$$

Taking the logarithm on both sides, gives Equation (10).

$$\ln p(\boldsymbol{x}, A) + \frac{N}{2}\ln 2\pi\sigma^2 = -\frac{1}{2\sigma^2}\sum_{n=0}^{N-1}(x[n] - A)^2 \quad (10)$$

Now, defining a new function of A, $f(A) = \ln p(\boldsymbol{x}, A) + \frac{N}{2}\ln 2\pi\sigma^2$, which is a ***shifted log-likelihood***, gives Equation (11).

$$f(A) = -\frac{1}{2\sigma^2}\sum_{n=0}^{N-1}(x[n] - A)^2 \quad (11)$$

with

$$f(0) = -\frac{1}{2\sigma^2}\sum_{n=0}^{N-1}x[n]^2 \quad (12)$$

$$\therefore f(0) = -\frac{N\bar{x}^2}{2\sigma^2}$$

Differentiating Equation (11) with respect to A, gives Equation (13).

$$f'(A) = \sum_{n=0}^{N-1}\left(\frac{1}{\sigma^2}x[n] - A\right) \quad (13)$$

$$\therefore f'(0) = \frac{N\bar{x}}{\sigma^2} \quad (14)$$

Suppose the Laplace transform of $f(A)$ with respect to A exist and is defined as Equation (15):

$$F(s) = \int_0^\infty f(A)e^{-sA}dA \quad (15)$$

Then from equation (11), with the linearity property of Laplace transformation, gives Equation (16).

$$F(s) = -\frac{N}{2\sigma^2}\left[\frac{\bar{x}^2}{s} - \frac{2\bar{x}}{s^2} + \frac{2}{s^3}\right] \quad (16)$$

From the properties of Laplace transform, we have Equation (17) and substituting $F(s), f(0)$, and $f'(0)$ gives Equation (18).

$$L\{f''(A)\} = s^2F(s) - sf(0) - f'(0) \quad (17)$$

$$L\{f''(A)\} = -\frac{N}{\sigma^2 s} \quad (18)$$

Taking inverse Laplace transform, gives Equation (19).

$$f''(A) = -\frac{N}{\sigma^2} \quad (19)$$

Intuitively, this is equal to the Fisher Information measure. Then, CRLB of A is defined as shown in Equation (20).

$$CRLB_{\hat{A}} = -\frac{1}{f''(A)} = -\frac{1}{L^{-1}[s^2F(s) - sf(0) - f'(0)]} \quad (20)$$

From the Equation (20), CRLB determination has been simplified notably, only algebraic manipulation of known functions is required, as summarized in Figure 1. Consequently, the MVUE can be readily obtained from the relation shown in Equation (21).

$$\hat{A} = -\frac{f'(0)}{f''(A)} = -\frac{f'(0)}{L^{-1}[s^2F(s) - sf(0) - f'(0)]} \quad (21)$$
$$\Rightarrow \hat{A} = \bar{x}$$

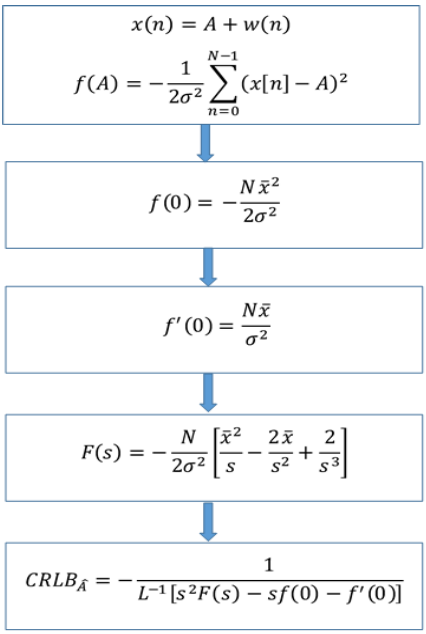


Figure 1: summary procedure to determine CRLB of a DC signal.

B. Generalization of CRLB for $\alpha = A^n$

Theorem 2

If CRLB of A is $CRLB_{\hat{A}}$ then, the CRLB of $\alpha = A^n$, for positive values of n will be:

$$CRLB_{\hat{\alpha}} = CRLB_{\hat{A}} \times L^{-1}\left(\frac{T[n]}{s^{2n-1}}\right) \quad ; n > 0 \quad (22)$$

$$T[n] = n^2 n! \quad (23)$$

The function $T[n]$ is termed as T-function and its values for the first five values of the positive integers, n, are summarized in Table 1.

Table 1: T-function values

n	1	2	3	4	5
T(n)	1	8	54	384	3000

Observations

Testing the theorem for $n = 1$, meaning $\alpha = A$ and compared with the existing approach.

A. Existing method

$$CRLB_{\hat{\alpha}} = \left(\frac{\partial \alpha}{\partial A}\right)^2 \times CRLB_{\hat{A}} \quad (24)$$

$$CRLB_{\hat{\alpha}} = CRLB_{\hat{A}} \quad (25)$$

Similarly, when $n = 2$, i.e. $\alpha = A^2$, again using (24), gives

$$CRLB_{\hat{\alpha}} = 4A^2 \times CRLB_{\hat{A}} \quad (26)$$

B. Proposed method

Using the T-function table, given in Table 1, gives $T[1] = 1$. Then, using Equation (22), yields

$$CRLB_{\hat{\alpha}} = CRLB_{\hat{A}} \times L^{-1}\left(\frac{1}{s}\right) \quad (27)$$

When using the table of Laplace transform (Schiff 1999) with t replaced by A, also gives (25). For $n = 2$, meaning $\alpha = A^2$. Again, using T-function table from Table 1 gives $T[2] = 8$. Again using Equation (22), gives

$$CRLB_{\hat{\alpha}} = CRLB_{\hat{A}} \times L^{-1}\left(\frac{8}{s^3}\right) \quad (28)$$

Using the Laplace transform table (Schiff 1999) with t replaced by A, gives:

$$L^{-1}\left(\frac{8}{s^3}\right) = 4A^2$$
$$\therefore CRLB_{\hat{\alpha}} = 4A^2 \times CRLB_{\hat{A}} \qquad (29)$$

In the proposed approach, only algebraic manipulations are needed to arrive at the same solution.

Theorem 3
If the CRLB of A, ($CRLB_{\hat{A}}$), is known, then the CRLB of α for negative values of n is shown by Equation (30).

$$CRLB_{\hat{\alpha}} = CRLB_{\hat{A}} \times \frac{L^{-1}\left(\frac{T[-n]}{s^{2|n|-1}}\right)}{L^{-1}\left(\frac{(4|n|)!}{s^{4|n|+1}}\right)} \quad ; n < 0 \qquad (30)$$

Observations
For $n = -1$, meaning $\alpha = A^{-1}$

A. With existing method
Using Equation (24), gives

$$\therefore CRLB_{\hat{\alpha}} = A^{-4} \times CRLB_{\hat{A}} \qquad (31)$$

Similarly, when $n = -2$, again, using Equation (22), we have

$$\left(\frac{\partial \alpha}{\partial A}\right)^2 = 4A^{-6}$$
$$\therefore CRLB_{\hat{\alpha}} = 4A^{-6} \times CRLB_{\hat{A}} \qquad (32)$$

B. With proposed method
Using Table 1, gives $T[1] = 1$. Then, using Equation (31), yields Equation (33). When using the table of Laplace transform given in Schiff (1999) with t replaced by A, also gives Equation (31). Then, from Equation (30), we have

$$CRLB_{\hat{\alpha}} = CRLB_{\hat{A}} \times \frac{L^{-1}\left(\frac{1}{s}\right)}{L^{-1}\left(\frac{4!}{s^5}\right)} \qquad (33)$$
$$\therefore CRLB_{\hat{\alpha}} = A^{-4} \times CRLB_{\hat{A}}$$

For $n = 2$, meaning $\alpha = A^2$. Again, using T-function table from Table 1 gives $T[2] = 8$ and therefore, Equation (30), gives

$$CRLB_{\hat{\alpha}} = CRLB_{\hat{A}} \times \frac{L^{-1}\left(\frac{8}{s^3}\right)}{L^{-1}\left(\frac{8!}{s^9}\right)} \qquad (34)$$
$$\therefore CRLB_{\hat{\alpha}} = 4A^{-6} \times CRLB_{\hat{A}}$$

The procedure to obtain the CRLB of a function DC signal, $\alpha = A^n$ for both positive and negative values of n is summarized in Figure 2.

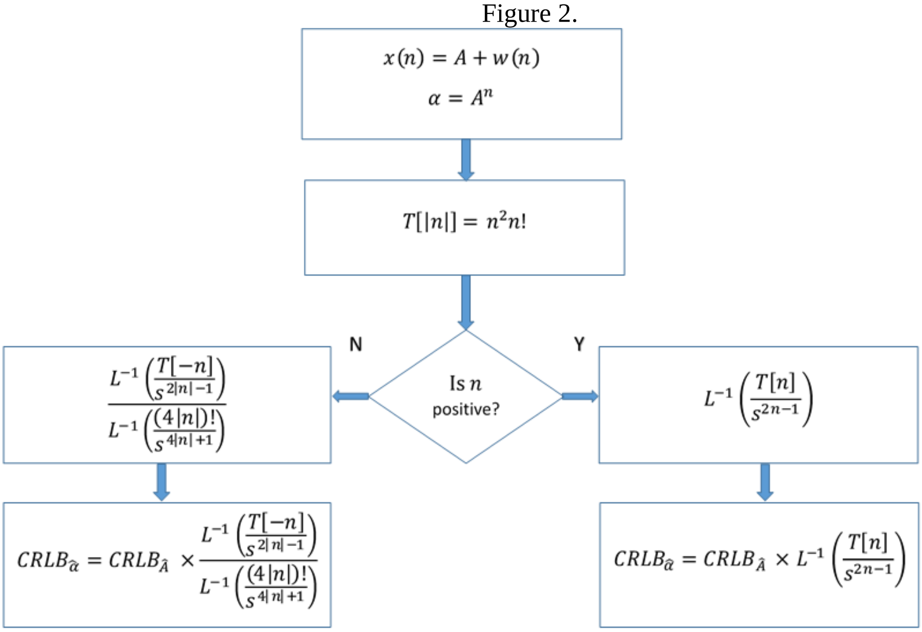


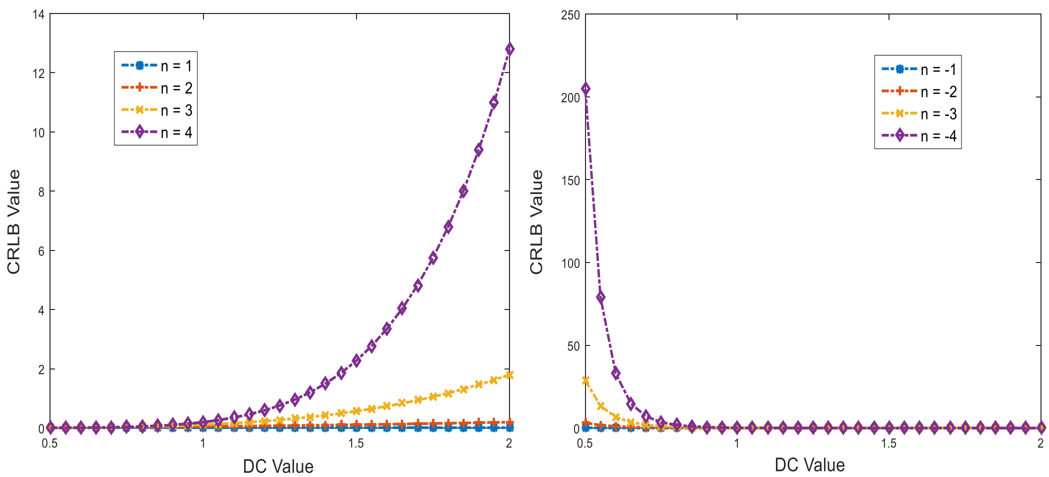
Figure 2: Summary procedure to determine CRLB of $\boldsymbol{\alpha = A^n}$.

Results Discussions

Though the proposed method has relatively long expressions compared to the existing ones, the two methods seem to have almost the same number of steps. Additionally, the latter requires only algebraic manipulations and look-up tables to arrive at the same solution as the existing method. As depicted in the previous section, all complex mathematical manipulations performed in the former approach have been mapped to simpler algebraic expressions making the proposed approach superior over the traditional method. With the new approach, only Laplace transform and T-function tables are used making the method easy to handle. Once the expression for the CRLB is obtained using Theorems 1-3, then MVUE can be deduced. Definitely, the idea presented by this paper will stimulate more research work in mathematical applications in engineering problem solving using the capability of Laplace transformation.

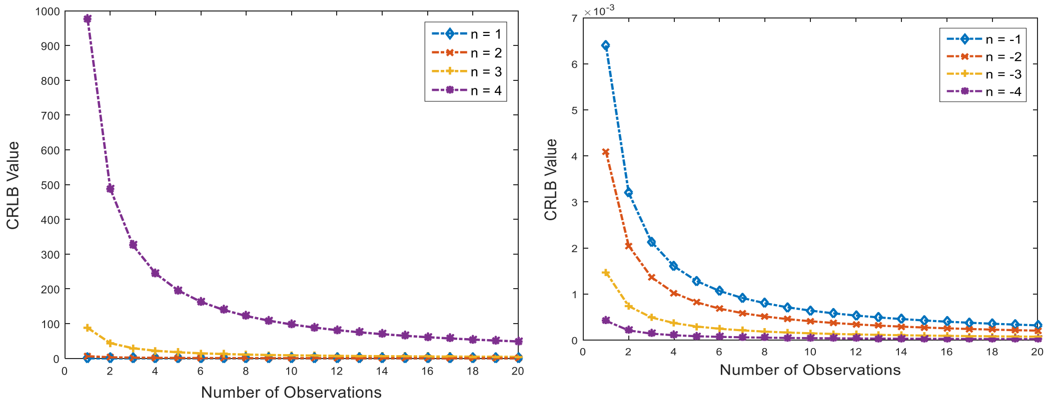
From the above observations, the CRLB was found to be degraded with the increasing DC values for a fixed number of observations in case of positive integer values of n as shown in Figure 3 (a). But, with negative integer values, the bound seems to be improved with increasing DC value as in Figure 3 (b). As the magnitude of integer value increases, the bound becomes worse than before.

However, the bound improves significantly if the integer value n is negative as depicted in Figure 3 (b). It is also evident that the bound improves as more and more observations are made keeping the estimated signal value constant for both cases: positive and negative integer values as shown in Figure 4 (a) and (b), respectively. Though, a better estimate is expected for large negative integers contrary to positive integers. The CRLB is even function of A, meaning the CRLB for negative values of A can easily be obtained by using positive range of A. That is, reflection through the vertical axis can be used to determine the CRLB in case of negative DC values. This reflecting behaviour of the CRLB makes the analysis sensible by avoiding the analysis of negative range of A.



(a) (b)

Figure 3: Variations of CRLB with DC values with (a) positive integer power (b) negative integer power.



(a) (b)

Figure 4: Variations of CRLB with the number of observations (a) positive integer values (b) negative integer values.

Conclusions

This work investigated the possibility of using Laplace transformation technique to determine the expressions of the Cramer-Rao Lower Bound (CRLB) and Minimum Variance Unbiased Estimator (MVUE) for the DC signal corrupted by AWGN. One advantage of the Laplace transformation is its ability to map all complicated mathematical manipulations into simple algebraic expressions. From the analysis, closed-form expressions for the determination of the CRLB and the MVUE by means of Laplace transformation were presented. Also, the more generalized forms of expressions for different values of integer powers of the DC value employing the Laplace transformation were developed. Contrary to the existing approach, the proposed alternative provides an easy and user-friendly method for the determination of the CRLB and MVUE for the DC signal under AWGN with very basic knowledge of mathematics required, as supported by Theorems 1-3.

References

Alslaimy M and Smith GE 2019 ATSC signal Cramér-Rao lower bound for emitter selection in passive radar systems. In *2019 IEEE Radar Conference RadarConf 2019* 1–6. IEEE, Boston.

Huang J, Gu K, Wang Y, Zhang T, Liang J, and Luo S 2020 Connectivity-based localization in ultra-dense networks: CRLB, theoretical variance, and MLE. *IEEE Access* 8: 35136–35149.

Hung NT, Crasta N, Moreno-Salinas D, Pascoal AM, and Johansen TA 2020 Range-based target localization and pursuit with autonomous vehicles: An approach using posterior CRLB and model predictive control. *Robot. Auton. Syst*. 132: 103608.

Kay S 1993 *Fundamentals of statistical signal processing, vol. I: estimation theory*. Prentice Hall, New Jersey.

Khorasani SM, Hodtani GA, and Kakhki MM 2020 Decreasing Cramer–Rao lower bound by preprocessing steps. *Signal Image Video Proces.* 14(4): 781–789.

Li J, Lu IT, and Lu J 2021 Cramer-Rao lower bound analysis of data fusion for fingerprinting localization in non-line-of-sight environments. *IEEE Access* 9: 18607–18624.

Mao C, Wang K, Shi J, and Wang X 2020 Cramér-Rao bounds on angle estimating in bistatic MIMO radar with non-orthogonal waveforms. *IEEE Access* 8: 62639–62649.

Phillips CL and Parr JM 2008 *Signals, systems and transforms* 4th ed. Pearson.

Schiff JL 1999 *The Laplace transform: theory and applications*. Springer, Verlag New York.

Tian T, Du X, and Li G 2020 Cramer-Rao bounds of localization estimation for integrated radar and communication system. *IEEE Access* 8: 105852–105863.

Williams J 1973 *Laplace transforms, problem solvers*. George Allen & Unwin.