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Application of Faulkenberry (2018) Bayes Factor to a Balanced Two Way Analysis of Variance with Random Effects

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Abstract

The Analysis of Variance technique estimates variance components by comparing their mean squares to their expected values. Nevertheless, this method could give variance component estimates that are found outside the parameter space, i.e. negative estimates. In a bid to overcome this deficiency, alternate approaches are essential, and likelihood-based approaches have become common over time. Bayesian techniques have also been proposed and Bayes factors developed for examining various models. We applied the Bayes factor proposed by Faulkenberry (2018) to a Balanced Two Way ANOVA under three (3) cases, namely **Case 1**: the levels of the two factors are fixed; **Case 2**: the levels of the two factors are random; and **Case 3**: the levels of one factor are considered as fixed, while the levels of the other factor are considered as random. We realized that when the levels of the two factors are fixed, the Bayesian conclusion about the variability in the effects is in line with that of a frequentist. But when the same data set was considered to be wholly or partly as sample observations drawn randomly from a given population of interest, the Bayesian conclusion differed slightly from that of the frequentist.

Keywords: Bayes Factor; Bayesian; Frequentist, Fixed; Random.

Introduction

An extensive range of techniques has been established over time for variance components estimations; some of these techniques are: Analysis of Variance (ANOVA), Restricted Maximum Likelihood (REML), Maximum Likelihood (ML) and the Bayesian techniques. The ANOVA technique estimates variance components by comparing their mean squares to their expected values. However, this method could give variance component estimates that are found beyond the parameter space, i.e. negative estimates. In a bid to overcome this deficiency, alternate approaches are essential and likelihood-based approaches have become commonly used (Searle et al. 1992, Theobald et al. 1997, Basar and Firat 2016). The Maximum Likelihood technique for estimating components of the variance does not give account for the loss of various degrees of freedom instigated by estimating the fixed effects in the model. The REML technique remains more advantageous than the ML technique in that; "it permits for several random factors in the model" and "often with regards to the variance, maximizes the part of the function of the likelihood that does not depend on fixed effects (Basar and Firat 2016).

Bayesian technique for estimating the variance components is an alternate approach to the likelihood based techniques stated above. Bayesian methods were viewed

unfavourably by many statisticians due to philosophical and practical considerations. Several Bayesian methodologies required a lot of computation to complete. However, with the advent of some powerful computers and newer algorithms like Markov Chain Monte Carlo (MCMC), Bayesian methods have seen increasing uses within statistics in the 21st century.

The Bayesian technique for hypothesis testing about variance component(s) is obtained by the means of the Bayes factor which is given by

$$BF_{01} = \frac{p(\boldsymbol{y}|M_0)}{p(\boldsymbol{y}|M_1)} \qquad (1)$$

The Bayes factor equates the marginal densities of the data $\boldsymbol{y}$ under the models, $M_0$and M_1. M_0 is the hypothesis that one or more of the components of the variance equals to zero, whereas M_1 is the hypothesis that the variance is unrestricted (Egburonu and Abidoye 2018b).

Carrying out Bayesian analysis requires a prior distribution. Faulkenberry (2018) developed a Bayesian Information Criterion (BIC)-based Bayes factor that can be obtained from table of ANOVA summaries. The computation of the Bayes factor requires a specification of prior. The Faulkenberry (2018) Bayes factor assumes an implicit choice of prior, one which is called the "Unit Information Prior" (Masson 2011). The Unit Information Prior (UIP) is a data-dependent prior, (typically multivariate Normal) with mean at the MLE, and precision equal to the information provided by one observation. A different choice of prior will result in a different value for the Bayes factor. However, it has been previously shown through simulations that this difference is marginal, and the result of the formula tends to be fairly consistent with other choices of prior (Faulkenberry 2017).

This paper illustrated an alternative Bayesian methodology by applying the Faulkenberry (2018) Bayes Factor to a balanced Two Way ANOVA with Random Effects under three (3) cases, namely: **Case 1**: the levels of the two factors are fixed; **Case 2**: the levels of the two factors are random; and **Case 3**: the levels of one factor (i.e. Factor A) are considered as fixed while the levels of the other factor (i.e. Factor B) are considered as random.

Materials and Methods

The two ways ANOVA model is given by:

$$y_{ijk} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij} + \varepsilon_{ijk} \quad \begin{cases} i = 1, 2, \dots, m \\ j = 1, 2, \dots, n \\ k = 1, 2, \dots, p \end{cases} \qquad (2)$$

Where

y_{ijk} is the kth observation in the ith row and the jth column,

μ is the grand mean,

α_i is the ith effect of factor A,

β_j is the jth effect of factor B,

$(\alpha\beta)_{ij}$ is the interaction effect of ith factor A and jth factor B, and, and

ε_{ijk} is error term associated with y_{ij} (residual) effects.

Considering a balanced variance components model indicated by Equation (2) above, the interests of Bayesians centre around assessing the random effects in order to ascertain whether it ought to be incorporated; this is comparable to testing the following null hypotheses:

1. Is there variability in the effects of the various levels of factor A?
2. Is there variability in the effects of the various levels of factor B?
3. Is there variability in the interaction effects between the various levels of Factors A and B?

Symbolically written as:

$$H_{01} : \sigma_\alpha^2 = 0 \; verses \; H_{11}: \sigma_\alpha^2 \neq 0 \qquad (3)$$

$$H_{02} : \sigma_\alpha^2 = 0 \; verses \; H_{12}: \sigma_\beta^2 \neq 0 \qquad (4)$$

$$H_{03} : \sigma_\alpha^2 = 0 \; verses \; H_{13}: \sigma_{\alpha\beta}^2 \neq 0 \qquad (5)$$

When the model in Equation 2 includes random effect(s), the Expected Mean Squares (EMS) will most often differ from the same model having fixed effect(s). Consequently, this will affect the manner in which the F-statistics are computed, its distribution and degrees of freedom.

When the model in Equation 2 includes fixed and random effects,

For the fixed factor A,

$$\sum_{i=1}^{m} \alpha_i = 0 \qquad (6)$$

For the random factor B,

$$\beta_j \sim iid\ N(0, \sigma_\beta^2)$$

For the interaction effect,

$$(\alpha\beta)_{ij} \sim iid \quad N(\ 0, \quad \frac{m-1}{m}\sigma_{\alpha\beta}^2)$$

Subject to the restriction:

$$\begin{cases} \sum_{i=1}^{m}(\alpha\beta)_{ij} = 0 \\ Cov\left((\alpha\beta)_{ij}, (\alpha\beta)_{i'j}\right) = -\frac{1}{m}\sigma_{\alpha\beta}^2 \quad \forall\ i \neq i' \end{cases} \qquad (7)$$

And

ε_{ijk},

β_j and $(\alpha\beta)_{ij}$ are pairwise independent

1. $E(y_{ijk}) = \mu + \alpha_i$ (8)
2. $Var(y_{ijk}) = \sigma^2 + \sigma_\beta^2 + \frac{m-1}{m}\sigma_{\alpha\beta}^2$ (9)

BIC-Based Bayes Factors Proposed By Faulkenberry (2018)

Based on work by Raftery (1995) and Wagenmakers (2007), Faulkenberry (2018) demonstrated a technique for estimating Bayes factors by means of the Bayesian Information Criterion (BIC).

For a given model H_i, the BIC is defined as,

$$BIC\ (H_i) = -2\ Log\ L_i + k_i\ Log\ z \qquad (10)$$

Where,

z is the number of observations,

k_i is the number of free parameters of model H_i,

L_i is the maximum likelihood for model H_i

(i.e. $L_i = c\ \Pr(D|\hat{\theta}, H_i)$ (11)

c is an arbitrary constant (Edwards 1992).

Wagenmakers (2007) showed that in the case of two models, H_0 and H_1, the Bayes factor is defined as the ratio of the prior predictive probabilities; hence, the BIC approximation of the Bayes factor is given by:

$$BF_{01} = \frac{Pr_{BIC}(D|H_0)}{Pr_{BIC}(D|H_1)} = exp\frac{\Delta BIC_{10}}{2} \qquad (12)$$

Where

$$\Delta BIC_{10} = BIC(H_1) - BIC(H_0) \qquad (13)$$

$$\Delta BIC_{10} = z\log\left(\frac{SSE_1}{SSE_0}\right) + (k_1 - k_0)\log z \qquad (14)$$

In the Equation (14) above, SSE_0 and SSE_1 represent the sum of squares for the error terms in models H_0 and H_1, respectively.

Both Wagenmakers (2007) and Masson (2011) gave examples of how to use this approximation to compute Bayes factors, assuming one is given information about, SSE_0 and SSE_1, as is the case with most statistical software. However, if one is only given the ANOVA summary (e.g., $F(1, 23) = 4.35$), the computation is nontrivial.

To begin, suppose we wish to examine an effect of some independent variable with associated $F - ratio\ F(df_1, df_2)$, where df_1 represents the degrees of freedom associated with the manipulation, and df_2 represents the degrees of freedom associated with the error term. Then,

$$F = \frac{\frac{SS_1}{df_1}}{\frac{SS_2}{df_2}} \qquad (15)$$

$$= \frac{SS_1}{SS_2} \times \frac{df_1}{df_2} \qquad (16)$$

where SS_1 and SS_2 are the sum of squared errors associated with the manipulation and the error term, respectively. From Equation (14), we see that

$$\Delta BIC_{10} = z\log\left(\frac{SSE_1}{SSE_0}\right) + (k_1 - k_0)\log z$$

$$= z\log\left(\frac{SS_2}{SS_1 + SS_2}\right) + (k_1 - k_0)\log z \qquad (17)$$

This equality holds because SSE_1 represents the sum of squares that is not explained by H_1, which is simply SS_2 (the error term). Similarly, SSE_0 is the sum of squares not explained by H_0, which is the sum of SS_1 and SS_2 (Wagenmaker 2007). Finally, in the context of comparing H_0 and H_1 in an ANOVA design, we have $(k_1 - k_0) = df_1$. Now, we can use algebra to re-express ΔBIC_{10} in terms of F:

$$\Delta BIC_{10} = z\log\left(\frac{SS_2}{SS_1 + SS_2}\right) + df_1\log z \qquad (18)$$

$$= z\log\left(\frac{1}{\frac{SS_1}{SS_2}+1}\right) + df_1\log z$$

$$= z\log\left(\frac{\frac{df_2}{df_1}}{\frac{SS_1}{SS_2}\times\frac{df_2}{df_1}+\frac{df_2}{df_1}}\right) + df_1\log z \quad = z\log\left(\frac{\frac{df_2}{df_1}}{F+\frac{df_2}{df_1}}\right) + df_1\log z$$

$$= z\log\left(\frac{df_2}{Fdf_1+df_2}\right) + df_1\log z \qquad (19)$$

Substituting Equation (19) into Equation (12), we can compute:

$$BF_{01} = exp\frac{\Delta BIC_{10}}{2}$$

$$= exp\left[\frac{1}{2}\left[zLog_e\left(\frac{df_2}{Fdf_1+df_2}\right) + df_1\,Log_e\,z\right]\right]$$

$$= exp\left[\frac{z}{2}\,Log_e\left(\frac{df_2}{Fdf_1+df_2}\right) + \frac{df_1}{2}Log_e\,z\right]$$

$$= \left(\frac{df_2}{Fdf_1+df_2}\right)^{\frac{z}{2}} \times z^{\frac{df_1}{2}}$$

$$= \left(\frac{(df_2)^z \times z^{df_1}}{(Fdf_1+df_2)^z}\right)^{\frac{1}{2}}$$

$$= \left(\frac{z^{df_1}}{\left(\frac{Fdf_1}{df_2}+1\right)^z}\right)^{\frac{1}{2}}$$

$$= \sqrt{\frac{z^{df_1}}{\left(\frac{Fdf_1}{df_2}+1\right)^z}}$$

$$= \sqrt{z^{df_1}\left(\frac{Fdf_1}{df_2}+1\right)^{-z}} \qquad (20)$$

The Bayes factor in Equation (20) compares favourably to Bayes factors computed using existing software solutions with raw data.

Note that,

$$BF_{10} = \frac{1}{BF_{01}} \qquad (21)$$

The formula can be used flexibly to measure evidence for either the null hypothesis or its corresponding alternative hypothesis, subject to the researcher's needs and interests. Conclusions on the Bayes factor values are drawn from the table (Table 1) provided by Raftery (1995).

Table 1: Decision Table for interpretation of computed Bayes factor

Bayes Factor (BF_{01})	Evidence for the null hypothesis (H_0)
$1-3$	Not worth more than just a mere mention
$3-10$	Substantial
$10-100$	Strong
>100	Decisive

Source: Raftery (1995).

The study was carried out in the following steps:

Steps:

1. For each case (1, 2 and 3), data was simulated for the sets **m, n** and **p** combination.
2. The frequentist Two Way ANOVA table summary was computed using the simulated data for each case.
3. The Faulkenberry (2018) BIC-based Bayes factor was computed using results in step 2 above.
4. The results were discussed.

Results and Discussion

Data were simulated with the aid of the R statistical computing software (version 3.5.2) from a standard normal population with $mean = 0$ and $variance = 1$ i.e. $N(\mu = 0, \sigma = 1)$. The simulation was carried out using random seed sets to enable easy replication. The random sample generated contains 125 random numbers clustered in 25 cells (5 rows and 5 columns). Each cell

contains 5 random numbers. This design is similar to a setup of the regular two way ANOVA with five replicates per cell. Table 2 was generated from the process illustrated above. As a matter of choice, we decided to use a two way design with 5 by 5 categories and 5 replicates per cell although other designs may be applicable provided the F-statistics and degree of freedom can be obtained through the frequentist technique (Faulkenberry 2018, Egburonu 2018, Egburonu and Abidoye 2018a, Egburonu and Abidoye 2018b). The choice of this design is to replicate experimental conditions that are common across many applications in the biological and behavioural sciences.

Table 2: Simulated data`

		Factor B				
		Level 1	**Level 2**	**Level 3**	**Level 4**	**Level 5**
Factor A	**Level 1**	-0.90, 0.18, 1.59, -1.13, -0.08	0.13, 0.71, -0.24, 1.98, -0.14	0.42, 0.98, -0.39, -1.04, 1.78	-2.31, 0.88, 0.04, 1.01, 0.43	2.09, -1.20, 1.59, 1.95, 0.00
	Level 2	-2.45, 0.48, -0.60, 0.79, 0.29	0.74, 0.32, 1.08, -0.28, -0.78	-0.60, -1.73, -0.90, -0.56, -0.25	-0.38, -1.96, -0.84, 1.90, 0.62	1.99, -0.31, -0.09, -0.18, -1.20
	Level 3	-0.84, 2.07, -0.56, 1.28, -1.05	-1.97, -0.32, 0.94, 1.14, 1.67	-1.79, 2.03, -0.70, 0.16, 0.51	-0.82, -2.00, -0.48, 0.08, -0.90	-0.92, 0.33, -0.14, 0.43, -0.05
	Level 4	-0.91, 1.30, 0.77, 1.05, -1.41	1.00, -1.70, -0.53, -1.37, -2.21	1.82, -0.65, -0.28, -0.39, 0.39	1.60, 1.68, -1.18, -1.36, -1.51	-1.25, 1.96, 0.01, -0.84, -0.86
	Level 5	1.07, 0.26, -0.31, -0.75, -0.35	2.05, 0.94, 2.01, -0.42, 0.56	-1.03, -0.25, 0.47, 1.36, -0.78	0.46, 1.23, 1.15, 0.11, -1.04	1.24, 0.14, 1.71, -0.43, -0.86

Source: Simulation results.

From Table 2,

$Number\ of\ rows\ (m) = 5$

$Number\ of\ columns\ (n) = 5$

$Number\ of\ observations\ per\ cell\ (p) = 5$

$$SST = \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{p}(y_{ijk} - \bar{Y}_{...})^2 = \sum_{i=1}^{5}\sum_{j=1}^{5}\sum_{k=1}^{5}(y_{ijk} - \bar{Y}_{...})^2 = 158.0611$$

$$SSA = np\sum_{i=1}^{m}(\bar{Y}_{i..} - \bar{Y}_{...})^2 = 5(5)\sum_{i=1}^{5}(\bar{Y}_{i..} - \bar{Y}_{...})^2 = 7.4092$$

$$SSB = mp\sum_{j=1}^{n}(\bar{Y}_{.j.} - \bar{Y}_{...})^2 = 5(5)\sum_{j=1}^{5}(\bar{Y}_{.j.} - \bar{Y}_{...})^2 = 2.0779$$

$$SSAB = p\sum_{i=1}^{m}\sum_{j=1}^{n}(\bar{Y}_{ij.} - \bar{Y}_{i..} - \bar{Y}_{.j.} + \bar{Y}_{...}) = 5\sum_{i=1}^{5}\sum_{j=1}^{5}(\bar{Y}_{ij.} - \bar{Y}_{i..} - \bar{Y}_{.j.} + \bar{Y}_{...}) = 14.5775$$

$$SSE = SST - SSA - SSB - SSAB = 158.0611 - 7.4092 - 2.0779 - 14.5775 = 133.9965$$

$$MSA = \frac{SSA}{m-1} = 1.8523$$

$$MSB = \frac{SSB}{n-1} = 0.5195$$

$$MSAB = \frac{SSB}{(m-1)(n-1)} = 0.9111$$

$$MSE = \frac{SSE}{mn(p-1)} = 1.3400$$

Analysis of data

Case 1: Levels of the Factor A and Levels of the Factor B are fixed

In this subsection (Case 1), we carried out a two way ANOVA with levels of both factors considered as fixed. The F-statistics obtained from the procedure were applied to the Faulkenberry (2018) Bayes factor and the results discussed.

Hypothesis testing for Case 1: Levels of the Factor A and Levels of the Factor B are fixed

The Faulkenberry (2018) Bayes factor is obtained using the F-statistics already computed in the Table 3:

$$H_{01} : \sigma_{\alpha}^2 = 0 \; against \; H_{11}: \sigma_{\alpha}^2 \neq 0 \quad (22)$$

The Faulkenberry (2018) Bayes factor is given by:

$$BF_{01} = \sqrt{z^{df_1}\left(\frac{Fdf_1}{df_2}+1\right)^{-z}}$$
$$= \sqrt{125^4\left(\frac{1.3823 \times 4}{100}+1\right)^{-125}}$$
$$= 540.725$$

The Bayes factor $BF_{01} = 540.725$, signifies that the data have a decisive evidence in favour of the stated null hypothesis of zero variation in the levels of factor A as indicated by Equation (22).

To test the hypothesis below,

$$H_{02} : \sigma_{\beta}^2 = 0 \; against \; H_{12}: \sigma_{\beta}^2 \neq 0 \quad (23)$$

The Faulkenberry (2018) Bayes factor is given by:

$$BF_{01} = \sqrt{z^{df_1}\left(\frac{Fdf_1}{df_2}+1\right)^{-z}}$$
$$= \sqrt{125^4\left(\frac{0.3877 \times 4}{100}+1\right)^{-125}}$$
$$= 5972.2009$$

The Bayes factor $BF_{01} = 5972.2009$, signifies that the data have a decisive evidence in favour of the stated null hypothesis of zero variation in the levels of factor B indicated by Equation (23). To test the hypothesis below

$$H_{03} : \sigma_{\alpha\beta}^2 = 0 \; against \; H_{13}: \sigma_{\alpha\beta}^2 \neq 0 \quad (24)$$

The Faulkenberry (2018) Bayes factor is given by:

$$BF_{01} = \sqrt{z^{df_1}\left(\frac{Fdf_1}{df_2}+1\right)^{-z}}$$
$$= \sqrt{125^{16}\left(\frac{0.6799 \times 16}{100}+1\right)^{-125}}$$
$$= 9.3799 \times 10^{13}$$

The Bayes factor $BF_{01} = 9.3799 \times 10^{13}$, signifies that the data have a decisive evidence in favour of the stated null hypothesis of zero variation in the interaction effects of the levels of factors A and B indicated by Equation (24).

Table 3: ANOVA summary Table for Case 1

Source of variation	Degree of freedom (DF)	Sum of squares (SS)	Mean squares (MS)	F-Ratio
Factor A	$5-1=4$	$SSA = 7.4092$	$MSA = \frac{SSA}{m-1} = 1.8523$	$F_1 = \frac{MSA}{MSE} = 1.3823$
Factor B	$5-1=4$	$SSB = 2.0779$	$MSB = \frac{SSB}{n-1} = 0.5195$	$F_2 = \frac{MSB}{MSE} = 0.3877$
Interaction AB	$(5-1)(5-1)=16$	$SSAB = 14.5775$	$MSAB = \frac{SSAB}{(m-1)(n-1)} = 0.9111$	$F_3 = \frac{MSAB}{MSE} = 0.6799$
Error	$5(5)(5-1) = 100$	$SSE = 133.9965$	$MSE = \frac{SSE}{mn(p-1)} = 1.3400$	
Total	$5(5)(5)-1 = 124$	$SST = 158.0611$		

Discussion for Case 1 (Levels of factors A and B are fixed)

As summarized in Table 4, we can see that the entire hypothesis examined under Case 1, showed decisive evidences in favour of the null hypothesis of no variability in the effects of the levels of the factors as well as in their interactions. A side examination of a frequentist conclusion from the F-statistic table at a 5% level of significance, $F_{4,16} = 3.01$ and $F_{16,100} = 1.75$ indicates that the stated null hypothesis of zero treatment effect is not rejected. This shows that for a two way balanced ANOVA model with fixed effects, the Bayesian as well as the frequentist conclusions are not differing. Although, contrary to the frequentist' procedure, the Bayes factor indexes the extent to which the observed data supports one hypothesis over another; this we can see in the computations of the respective $BF_{10} = \frac{1}{BF_{01}}$. In this instance,

For hypothesis H_{01}, $BF_{10} = \frac{1}{540.725} = 1.85 \times 10^{-3}$ indicating very negligible support of the data for the alternative hypothesis.

For hypothesis H_{02}, $BF_{10} = \frac{1}{5972.2009} = 1.67 \times 10^{-4}$ indicating very negligible support of the data for the alternative hypothesis.

For hypothesis H_{03}, $BF_{10} = \frac{1}{9.3799 \times 10^{13}} = 1.07 \times 10^{-14}$ indicating very negligible support of the data for the alternative hypothesis.

The frequentist' procedure is less informative with respect to how the data supports the alternative hypothesis. The Bayesian inference can be thought of as a data-driven process for updating our belief in a hypothesis. The Bayes factor allows us a relative plausibility of comparing the competing hypothesis.

Table 4: Summary Table for Case 1

Summary Table for Case 1			
Hypothesis	F-Statistics	Faulkenberry (2018) Bayes factor (BF_{01})	Bayes factor in support of the alternative hypothesis (BF_{10})
$H_{01} : \sigma_\alpha^2 = 0 \; against \; H_{11}: \sigma_\alpha^2 \neq 0$	1.3823	540.725	1.85×10^{-3}
$H_{02} : \sigma_\beta^2 = 0 \; against \; H_{12}: \sigma_\beta^2 \neq 0$	0.3877	5972.2009	1.67×10^{-4}
$H_{03} : \sigma_{\alpha\beta}^2 = 0 \; against \; H_{13}: \sigma_{\alpha\beta}^2 \neq 0$	0.6799	9.3799×10^{13}	1.07×10^{-14}

Case 2: Levels of the Factor A and Levels of the Factor B are random.

In this subsection (Case 2), we carried out a two way ANOVA with levels of both factors considered as random. The F-statistics obtained from the procedure were applied to the Faulkenberry (2018) Bayes factor and the results discussed.

Hypothesis testing for Case 2: Levels of the Factor A and Levels of the Factor B are random

To test the hypothesis of interest, we obtain the Faulkenberry (2018) Bayes factor for each of the hypothesis using the F-statistics already computed in the Table 5:

$H_{01} : \sigma_\alpha^2 = 0 \; against \; H_{11}: \sigma_\alpha^2 \neq 0$ (25)

The Faulkenberry (2018) Bayes factor is given by:

$$BF_{01} = \sqrt{z^{df_1}\left(\frac{Fdf_1}{df_2} + 1\right)^{-z}}$$

$$= \sqrt{125^4\left(\frac{2.0330 \times 4}{16} + 1\right)^{-125}}$$

$$= 1.0946 \times 10^{-7}$$

The Bayes factor $BF_{01} = 1.0946 \times 10^{-7}$, signifies that the data has a very negligible evidence in favour of the stated null hypothesis of zero variations in the effects of the levels of factor A indicated by Equation (25). This implies that the data strongly favours the alternative hypothesis.

To test the hypothesis below,

$$H_{02}\ :\ \sigma_{\beta}^{2} = 0\ against\ H_{12}\colon \sigma_{\beta}^{2} \neq 0 \quad (26)$$

The Faulkenberry (2018) Bayes factor is given by:

$$BF_{01} = \sqrt{z^{df_1}\left(\frac{Fdf_1}{df_2}+1\right)^{-z}}$$
$$= \sqrt{125^{4}\left(\frac{0.5702\times 4}{16}+1\right)^{-125}}$$
$$= 3.7721$$

The Bayes factor $BF_{01} = 3.7721$, signifies that the data has a substantial (considerable) evidence in favour of the stated null hypothesis of zero variation in the effects of the levels of factor B indicated by Equation (26).

Table 5: ANOVA Summary Table for Case 2

Source of Variation	DF	SS	MS	F-Ratio
Factor A	$5-1=4$	$SSA = 7.4092$	$MSA = \frac{SSA}{m-1} = 1.8523$	$F_1 = \frac{MSA}{MSAB} = 2.0330$
Factor B	$5-1=4$	$SSB = 2.0779$	$MSB = \frac{SSB}{n-1} = 0.5195$	$F_2 = \frac{MSB}{MSAB} = 0.5702$
Interaction AB	$(5-1)(5-1)=16$	$SSAB = 14.5775$	$MSAB = \frac{SSAB}{(m-1)(n-1)} = 0.9111$	$F_3 = \frac{MSAB}{MSE} = 0.6799$
Error	$5(5)(5-1)=100$	$SSE = 133.9965$	$MSE = \frac{SSE}{mn(p-1)} = 1.3400$	
Total	$5(5)(5)-1=124$	$SST = 158.0611$		

To test the hypothesis below

$$H_{03}\ :\ \sigma_{\alpha\beta}^{2} = 0\ against\ H_{13}\colon \sigma_{\alpha\beta}^{2} \neq 0 \quad (27)$$

The Faulkenberry (2018) Bayes factor is given by:

$$BF_{01} = \sqrt{z^{df_1}\left(\frac{Fdf_1}{df_2}+1\right)^{-z}}$$
$$= \sqrt{125^{16}\left(\frac{0.6799\times 16}{100}+1\right)^{-125}}$$
$$= 9.3799\times 10^{13}$$

The Bayes factor $BF_{01} = 9.3799\times 10^{13}$, signifies that the data have a very decisive evidence in favour of the null hypothesis of zero variation in the interaction effects of various levels of the factors A and B indicated by Equation (27).

Discussion for Case 2 (Levels of factors A and B are random)

In testing the hypothesis $H_{01}\colon \sigma_{\alpha}^{2} = 0\ against\ H_{11}\colon \sigma_{\alpha}^{2} \neq 0$ in Equation (25), we realized (as summarized in Table 6) that the data showed very negligible (not worthy of mentioning) evidence in favour of the stated null hypothesis of no variation in the effects of the levels of the factor A. In fact, its inverse, $BF_{10} = \frac{1}{1.0946\times 10^{-7},} = 9.1358\times 10^{6}$ shows that the data have a very decisive evidence in favour of the alternative hypothesis that states that "there are variability in the effects of the levels of factor A". The test for the treatment effects in the levels of factor B and the interaction effects of the various levels of factor A and factor B indicated no variability. A side examination of a frequentist conclusion from the F-statistic

table at a 5% level of significance, $F_{4,16} = 3.01$ and $F_{16,100} = 1.75$ indicated that a stated null hypothesis of zero treatment effects is not rejected. However, we have seen from the Bayesian perspective that data showed very decisive evidence in favour of variability in effects of the levels of factor A. hence, a slightly differing opinion from that of the frequentist.

Case 3: The levels of one factor (i.e. Factor A) are considered as fixed, while the levels of the other factor (i.e. Factor B) are considered as random. (i.e. typical of a mixed effect model)

In this subsection (Case 3), we carried out a two way ANOVA with levels of factor A considered as fixed, while the levels of factor B are considered as random. The F-statistics obtained from the procedure were applied to the Faulkenberry (2018) Bayes factor and the results discussed.

Table 6: Summary Table for Case 2

Hypothesis	F-Statistics	Faulkenberry (2018) Bayes Factor (BF_{01})	Bayes factor in support of the alternative hypothesis (BF_{10})
$H_{01}: \sigma_\alpha^2 = 0 \; against \; H_{11}: \sigma_\alpha^2 \neq 0$	2.0330	1.0946×10^{-7}	9.1358×10^{6}
$H_{02}: \sigma_\beta^2 = 0 \; against \; H_{12}: \sigma_\beta^2 \neq 0$	0.5702	3.7721	0.2651
$H_{03}: \sigma_{\alpha\beta}^2 = 0 \; against \; H_{13}: \sigma_{\alpha\beta}^2 \neq 0$	0.6799	9.3799×10^{13}	1.07×10^{-14}

Table 7: ANOVA Summary Table for Case 3

Source of variation	DF	SS	MS	F-Ratio
Factor A	$5-1=4$	$SSA = 7.4092$	$MSA = \frac{SSA}{m-1} = 1.8523$	$F_1 = \frac{MSA}{MSAB} = 2.0330$
Factor B	$5-1=4$	$SSB = 2.0779$	$MSB = \frac{SSB}{n-1} = 0.5195$	$F_2 = \frac{MSB}{MSE} = 0.3877$
Interaction AB	$(5-1)(5-1) = 16$	$SSAB = 14.5775$	$MSAB = \frac{SSAB}{(m-1)(n-1)} = 0.9111$	$F_3 = \frac{MSAB}{MSE} = 0.6799$
Error	$5(5)(5-1) = 100$	$SSE = 133.9965$	$MSE = \frac{SSE}{mn(p-1)} = 1.3400$	
Total	$5(5)(5)-1 = 124$	$SST = 158.0611$		

Hypothesis testing for Case 3: Levels of Factor A are fixed while Levels of Factor B are random

To test the hypothesis of interest, we obtain the Faulkenberry (2018) Bayes factor for each of the hypothesis using the F-statistics already computed in the Table 7:

$$H_{01}: \sigma_\alpha^2 = 0 \; against \; H_{11}: \sigma_\alpha^2 \neq 0 \quad (28)$$

The Faulkenberry (2018) Bayes factor is given by:

$$BF_{01} = \sqrt{z^{df_1}\left(\frac{Fdf_1}{df_2}+1\right)^{-z}}$$
$$= \sqrt{125^4\left(\frac{2.0330 \times 4}{16}+1\right)^{-125}}$$
$$= 1.0946 \times 10^{-7}$$

The Bayes factor $BF_{01} = 1.0946 \times 10^{-7}$, signifies that the data have a very negligible evidence in favour of the above stated null hypothesis of zero variation in the effects of the levels of factor A indicated by Equation (28). This implies that the data very

decisively favoured the alternative hypothesis.

To test the hypothesis below,

$$H_{02} : \sigma_{\beta}^{2} = 0 \; against \; H_{12}: \sigma_{\beta}^{2} \neq 0 \quad (29)$$

The Faulkenberry (2018) Bayes factor is given by:

$$BF_{01} = \sqrt{z^{df_1}\left(\frac{Fdf_1}{df_2}+1\right)^{-z}}$$

$$= \sqrt{125^{4}\left(\frac{0.3877 \times 4}{100}+1\right)^{-125}}$$

$$= 5972.2009$$

The Bayes factor $BF_{01} = 5972.2009$, signifies that the data have a very decisive evidence in favour of the above stated null hypothesis of zero variation in the effects of the levels of factor B indicated by Equation (29).

To test the hypothesis below

$$H_{03} : \sigma_{\alpha\beta}^{2} = 0 \; against \; H_{13}: \sigma_{\alpha\beta}^{2} \neq 0 \quad (30)$$

The Faulkenberry (2018) Bayes factor is given by:

$$BF_{01} = \sqrt{z^{df_1}\left(\frac{Fdf_1}{df_2}+1\right)^{-z}}$$

$$= \sqrt{125^{16}\left(\frac{0.6799 \times 16}{100}+1\right)^{-125}}$$

$$= 9.3799 \times 10^{13}$$

The Bayes factor $BF_{01} = 9.3799 \times 10^{13}$, signifies that the data have very decisive evidence in favour of the null hypothesis of zero variation in the interaction effects of levels of factors A and B indicated by Equation (30).

Discussion for Case 3 (Levels of Factor A are fixed, while Levels of Factor B are random)

In testing the hypothesis $H_{01} : \sigma_{\alpha}^{2} = 0 \; against \; H_{11}: \sigma_{\alpha}^{2} \neq 0$, indicated by Equation (28), we realized that the data showed a very negligible evidence in favour of the stated null hypothesis of zero variation in the effects of levels of factor A. In fact, its inverse, $BF_{10} = \frac{1}{1.0946 \times 10^{-7},} = 9.1358 \times 10^{6}$ shows that the data have very decisive evidence in favour of the alternative hypothesis that states that "there are variability in the effects of the levels of the factor A". From Table 8, we can see also that the test for the treatment effects in levels of factor B indicated no effects variability so also the interaction effects AB. From the frequentist perspective, the F-statistic table at a 5% level of significance, $F_{4,16} = 3.01$ and $F_{16,100} = 1.75$ indicates that the null hypothesis of zero treatment effect is not rejected in all the three hypothesis tested. But we have seen from the Bayesian perspective that data showed a strong favour of variability in the effects of the levels in factor A; hence, a slightly differing opinion from that of the frequentist.

Real data illustration

In the **Appendix** below, real data on the efficiencies of Transfer Machines collected over a three weeks period was used to demonstrate the methodology discussed in this paper. The results obtained corroborated with that discovered using the simulated data in all the three cases.

Table 8: Summary Table for Case 3

Summary Table for Case 3			
Hypothesis	F-Statistics	Faulkenberry (2018) Bayes factor (BF_{01})	Bayes factor in support of the alternative hypothesis (BF_{10})
$H_{01} : \sigma_{\alpha}^{2} = 0 \; against \; H_{11}: \sigma_{\alpha}^{2} \neq 0$	2.0330	1.0946×10^{-7}	9.1358×10^{6}
$H_{02} : \sigma_{\beta}^{2} = 0 \; against \; H_{12}: \sigma_{\beta}^{2} \neq 0$	0.3877	5972.2009	1.6744×10^{-4}
$H_{03} : \sigma_{\alpha\beta}^{2} = 0 \; against \; H_{13}: \sigma_{\alpha\beta}^{2} \neq 0$	0.6799	9.3799×10^{13}	1.07×10^{-14}

Conclusion

In all the three cases studied in this paper (using both simulated and real data), it is evident that for the two way balanced ANOVA model, when the levels of the two factors are considered fixed, the Bayesian conclusions about variations in the effects of the levels of the factors is in line with that of the frequentist. But when the same data set was considered to be wholly or partly as a random sample from a population, the Bayesian conclusion differed slightly from that of a frequentist. Of a certainty we recommend strongly that when carrying out hypothesis testing, the frequentist as well as the Bayesian procedures should be employed because the Bayesian technique provides the researcher information on how the data supports the competing hypothesis. This information can guide the researcher in taking decisions. If one has raw data available, the options for computing Bayes factors are plentiful. But the Bayes factors espoused in this paper work even without having seen the raw data, which is helpful when judging evidential value of published results where contrasts are not pre-specified. It is a recommended methodology for a first introduction to Bayes factors, as it is a relatively simple calculation that comes directly from summary statistics. The Bayes factor demonstrated in this paper, although very convenient for computations, provides very large values for the Bayes factors in some instances. Further studies could focus on transformations of the Bayes factor so that it will offer smaller values.

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APPENDIX

Real-data application

The data below were collected on the efficiencies of transfer machines over a three week period.

Table AP 1: Efficiencies of transfer machines over a three week period

		Transfer machines		
Weeks		Transfer 1	Transfer 2	Transfer 3
	1	46, 37, 46	35, 40, 36	52, 50, 43
	2	46, 35, 47	63, 64, 60	52, 62, 56
	3	60, 47, 58	84, 68, 78	58, 47, 54

From **Table AP 1** above,

$Number\ of\ rows\ (m) = 3$

$Number\ of\ Columns\ (n) = 3$

$Number\ of\ Observations\ per\ cell\ (p) = 3$

$$SST = \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{p}(y_{ijk} - \bar{Y}_{...})^2 = \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}(y_{ijk} - \bar{Y}_{...})^2 = 3997.19$$

$$SSA = np\sum_{i=1}^{m}(\bar{Y}_{i..} - \bar{Y}_{...})^2 = 3(3)\sum_{i=1}^{3}(\bar{Y}_{i..} - \bar{Y}_{...})^2 = 1604.52$$

$$SSB = mp\sum_{j=1}^{n}(\bar{Y}_{.j.} - \bar{Y}_{...})^2 = 3(3)\sum_{j=1}^{3}(\bar{Y}_{.j.} - \bar{Y}_{...})^2 = 624.30$$

$$SSAB = p\sum_{i=1}^{m}\sum_{j=1}^{n}(\bar{Y}_{ij.} - \bar{Y}_{i..} - \bar{Y}_{.j.} + \bar{Y}_{...}) = 3\sum_{i=1}^{3}\sum_{j=1}^{3}(\bar{Y}_{ij.} - \bar{Y}_{i..} - \bar{Y}_{.j.} + \bar{Y}_{...}) = 1217.04$$

$$SSE = SST - SSA - SSB - SSAB = 551.33$$

$$MSA = \frac{SSA}{m-1} = 802.6$$

$$MSB = \frac{SSB}{n-1} = 312.15$$

$$MSAB = \frac{SSB}{(m-1)(n-1)} = 304.26$$

$$MSE = \frac{SSE}{mn(p-1)} = 30.63$$

Analysis of Data

Case 1: Levels of the two factors (i.e. Weeks and Transfer machines) are fixed
Table AP 2: ANOVA summary Table for Case 1

Source of variation	Degree of freedom (DF)	Sum of squares (SS)	Mean squares (MS)	F-Ratio	F-Critical
Weeks	$3-1=2$	$SSA = 1604.52$	$MSA = \frac{SSA}{m-1} = 802.26$	$F_1 = \frac{MSA}{MSE} = 26.19$	3.56
Transfer machines	$3-1=2$	$SSB = 624.30$	$MSB = \frac{SSB}{n-1} = 312.15$	$F_2 = \frac{MSB}{MSE} = 10.19$	3.56
Interaction	$(3-1)(3-1)=4$	$SSAB = 1217.04$	$MSAB = \frac{SSAB}{(m-1)(n-1)} = 304.26$	$F_3 = \frac{MSAB}{MSE} = 9.93$	2.93
Error	$3(3)(3-1) = 18$	$SSE = 551.33$	$MSE = \frac{SSE}{mn(p-1)} = 30.63$		
Total	$3(3)(3)-1 = 26$	$SST = 3997.19$			

Hypothesis testing for Case 1: Levels of the two factors are fixed
The Faulkenberry (2018) Bayes factor for the three hypotheses is computed as follows using the F-Ratio already computed in the Table AP 2 above:

$$H_{01}\ :\ \sigma_\alpha^2 = 0\ against\ H_{11}\colon \sigma_\alpha^2 \neq 0 \qquad (1)$$

$$BF_{01} = \sqrt{z^{df_1}\left(\frac{Fdf_1}{df_2}+1\right)^{-z}} = \sqrt{27^2\left(\frac{26.19\times 2}{18}+1\right)^{-27}} = 2.74\times 10^{-7}$$

$$H_{02}\ :\ \sigma_\beta^2 = 0\ against\ H_{12}\colon \sigma_\beta^2 \neq 0 \qquad (2)$$

$$BF_{01} = \sqrt{z^{df_1}\left(\frac{Fdf_1}{df_2}+1\right)^{-z}} = \sqrt{27^2\left(\frac{10.19\times 2}{18}+1\right)^{-27}} = 9.82\times 10^{-4}$$

$$H_{03}\ :\ \sigma_{\alpha\beta}^2 = 0\ against\ H_{13}\colon \sigma_{\alpha\beta}^2 \neq 0 \qquad (3)$$

$$BF_{01} = \sqrt{z^{df_1}\left(\frac{Fdf_1}{df_2}+1\right)^{-z}} = \sqrt{27^4\left(\frac{9.93\times 4}{18}+1\right)^{-27}} = 1.07\times 10^{-4}$$

Discussion for Case 1 (Levels of the two factors are fixed)
In all the three hypotheses examined under Case 1, there are very negligible evidences that the observed data favoured the stated null hypotheses. A side examination of a frequentist conclusion from the F-statistic table at a 5% level of significance indicated that the stated null hypothesis of zero treatment effect is not accepted. This shows that for a two way balanced ANOVA model with fixed effects, the Bayesian as well as the frequentist conclusions are not differing. This corroborates the conclusion from our earlier simulation illustrations above.

Case 2: Levels of the two factors are random

Table AP 3: ANOVA summary Table for Case 2

Source of variation	Degree of freedom (DF)	Sum of squares (SS)	Mean squares (MS)	F-Ratio	F-Critical
Weeks	$3-1=2$	$SSA = 1604.52$	$MSA = \frac{SSA}{m-1} = 802.26$	$F_1 = \frac{MSA}{MSAB} = 2.64$	6.94
Transfer Machines	$3-1=2$	$SSB = 624.30$	$MSB = \frac{SSB}{n-1} = 312.15$	$F_2 = \frac{MSB}{MSAB} = 1.03$	6.94
Interaction	$(3-1)(3-1)=4$	$SSAB = 1217.04$	$MSAB = \frac{SSAB}{(m-1)(n-1)} = 304.26$	$F_3 = \frac{MSAB}{MSE} = 9.93$	2.93
Error	$3(3)(3-1) = 18$	$SSE = 551.33$	$MSE = \frac{SSE}{mn(p-1)} = 30.63$		
Total	$3(3)(3)-1 = 26$	$SST = 3997.19$			

Hypothesis testing for Case 2: Levels of the two factors are random

The Faulkenberry (2018) Bayes factor for the three hypotheses is computed as follows using the F-Ratio already computed in the Table AP 3 above:

$$H_{01} : \sigma_\alpha^2 = 0 \; against \; H_{11}: \sigma_\alpha^2 \neq 0 \qquad (4)$$

$$BF_{01} = \sqrt{z^{df_1}\left(\frac{Fdf_1}{df_2}+1\right)^{-z}} = \sqrt{27^2\left(\frac{2.64\times 2}{4}+1\right)^{-27}} = 3.14\times 10^{-4}$$

$$H_{02} : \sigma_\beta^2 = 0 \; against \; H_{12}: \sigma_\beta^2 \neq 0 \qquad (5)$$

$$BF_{01} = \sqrt{z^{df_1}\left(\frac{Fdf_1}{df_2}+1\right)^{-z}} = \sqrt{27^2\left(\frac{1.03\times 2}{4}+1\right)^{-27}} = 0.10$$

$$H_{03} : \sigma_{\alpha\beta}^2 = 0 \; against \; H_{13}: \sigma_{\alpha\beta}^2 \neq 0 \qquad (6)$$

$$BF_{01} = \sqrt{z^{df_1}\left(\frac{Fdf_1}{df_2}+1\right)^{-z}} = \sqrt{27^4\left(\frac{9.93\times 4}{18}+1\right)^{-27}} = 1.07\times 10^{-4}$$

Discussion for Case 2 (Levels of the two factors are random)

In all the three hypotheses examined under Case 2, there are very negligible evidences from the Bayes factors computed that the observed data favoured the null hypothesis. A side examination of a frequentist conclusion from the F-statistics in Table AP 3, we could see difference in their conclusions. The frequentist did not reject the stated null hypothesis of zero variability in the effects of the levels of the Weeks and the Transfer machines. It only rejected that of the interaction effects; hence a differing conclusion from that of the Bayesian.

Case 3: Levels of the Weeks are fixed, while levels of the Machine transfers are random (i.e. typical of a mixed effect model)

Table AP 4: ANOVA summary Table for Case 3

Source of variation	Degree of freedom (DF)	Sum of squares (SS)	Mean squares (MS)	F-Ratio	F-Critical
Weeks	$3-1=2$	$SSA = 1604.52$	$MSA = \frac{SSA}{m-1} = 802.26$	$F_1 = \frac{MSA}{MSAB} = 2.64$	6.94
Transfer machines	$3-1=2$	$SSB = 624.30$	$MSB = \frac{SSB}{n-1} = 312.15$	$F_2 = \frac{MSB}{MSE} = 10.19$	3.56
Interaction	$(3-1)(3-1)=4$	$SSAB = 1217.04$	$MSAB = \frac{SSAB}{(m-1)(n-1)} = 304.26$	$F_3 = \frac{MSAB}{MSE} = 9.93$	2.93
Error	$3(3)(3-1)=18$	$SSE = 551.33$	$MSE = \frac{SSE}{mn(p-1)} = 30.63$		
Total	$3(3)(3)-1=26$	$SST = 3997.19$			

Hypothesis testing for Case 3 (Levels of weeks are fixed while levels of machine transfers are random)

The Faulkenberry (2018) Bayes factor for the three hypotheses is computed as follows using the F-Ratio already computed in the Table AP 4:

$H_{01}: \sigma_\alpha^2 = 0 \; against \; H_{11}: \sigma_\alpha^2 \neq 0$ (7)

$$BF_{01} = \sqrt{z^{df_1}\left(\frac{Fdf_1}{df_2}+1\right)^{-z}} = \sqrt{27^2\left(\frac{2.64\times 2}{4}+1\right)^{-27}} = 3.14\times 10^{-4}$$

$H_{02}: \sigma_\beta^2 = 0 \; against \; H_{12}: \sigma_\beta^2 \neq 0$ (8)

$$BF_{01} = \sqrt{z^{df_1}\left(\frac{Fdf_1}{df_2}+1\right)^{-z}} = \sqrt{27^2\left(\frac{10.19\times 2}{18}+1\right)^{-27}} = 9.82\times 10^{-4}$$

$H_{03}: \sigma_{\alpha\beta}^2 = 0 \; against \; H_{13}: \sigma_{\alpha\beta}^2 \neq 0$ (9)

$$BF_{01} = \sqrt{z^{df_1}\left(\frac{Fdf_1}{df_2}+1\right)^{-z}} = \sqrt{27^4\left(\frac{9.93\times 4}{18}+1\right)^{-27}} = 1.07\times 10^{-4}$$

Discussion for Case 3 (Levels of the Weeks are fixed, while levels of the Machine Transfers are random)

In all the three hypotheses examined under Case 3, there are very negligible evidences from the Bayes factors computed that the observed data favoured the stated null hypothesis. A side examination of a frequentist conclusion from the F-statistics in Table AP 4 indicates that there is a differing conclusion in that only the null hypothesis of zero variability in the effects of the weeks is not rejected.

Summary of Discussions (Real data application)

In all the 3 cases studied using real data, it is evident that for the two way balanced ANOVA model, when levels of the two factors are fixed, the Bayesian conclusions about variations in effects of levels of the factors are in line with that of the frequentist. But when the same data set was considered to be wholly or partly as a random sample from a population, the Bayesian conclusion differed slightly from that of the frequentist. This corroborates our earlier deductions from the use of simulated data as has been elaborately expressed in this paper.