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Semi-Analytic Approach to Solving Rosenau-Hyman and Korteweg-De Vries Equations Using Integral Transform

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Abstract

In this research, we proposed the fusing of Elzaki transform and projected differential transform (PDTM) to obtain an analytical or approximate solution of the Rosenau-Hyman and Korteweg-de Vries equations which respectively govern pattern formation in liquid drops and model of waves on shallow water surfaces. The results obtained presented in tables and graphs showed better efficiency, accuracy, and convergence of the method to handle Rosenau-Hyman and Korteweg-de Vries equations when compared to other methods in the literature.

Keywords: Rosenau-Hyman Equation; Korteweg-de Vries equation; Elzaki Projected differential transform method; Semi-analytic approach.

Introduction

For a long time, differential equations have played fundamental roles in all aspects of applied mathematics, and their relevance has grown with the introduction of the computer (Agbomola and Loyinmi 2022, Akinfe and Loyinmi 2022). Hence, the examination and analysis of differential equations cruising in applications resulted in many complex mathematical difficulties (Elzaki 2011, Lawal et al. 2017, Loyinmi et al. 2017a).

Due to the complexity of nature, practically all processes and phenomena in science and engineering are intrinsically nonlinear, and they are represented by nonlinear partial differential equations because there are several conditions and factors to consider in the system (Loyinmi and Akinfe 2020a, Akinfe and Loyinmi 2021, Erinle-Ibrahim et al. 2021). For this reason, nonlinear partial differential equations have piqued the interest of many mathematicians and applied scientists (Lawal and Loyinmi 2011a, b, Akinfe and Loyinmi 2020, Lawal and Loyinmi 2019, Akinfe and Loyinmi 2021). Nonlinear partial differential equations are frequently used to describe a wide range of processes and real-world phenomena, such as genetic configurations, mutation, and variations in Fisher's equation, magnetic flux, intensity, and quantum field theory in sine Gordon equation, shallow water waves and patterns in Korteweg-De-Vries (KdV) equation, advection-diffusion mechanisms and dynamics in Burgers-equation, Huxley's and so on (Loyinmi and Lawal 2011, Loyinmi and Oredein 2011, Lawal et al. 2017, Loyinmi and Akinfe 2020b, Babajide and Oluwatobi 2021).

Several studies have been carried out on differential equations (linear and nonlinear) over the years (Miura et al. 1968, Jang 2010, Dehghan et al. 2012, Loyinmi et al. 2017b, Lawal et al. 2018, Lawal et al. 2019a, Loyinmi and Akinfe 2020, Morenikeji et al. 2021). However, the need to provide convenient and consistent methods of the solution remains constant. Due to the difficulty in solving the exact solutions of nonlinear differential equations, semi-analytic solutions and numerical solutions are provided (Loyinmi et al. 2021, Oluwatobi and Erinle-Ibrahim 2021). It is however, important to consider the consistency and accuracy of such methods.

Researchers have combined the Elzaki transform and the projected differential transform in solving nonlinear partial differential equations (Elzaki et al. 2012, Elzaki and Alamri 2014, Suleman et al. 2017). The Elzaki Projected Differential Transform Method was first used in 2018 (Lu et al. 2018, Suleman et al. 2018). Afterward, the method was used to solve the generalized Burgers-Fisher's equation (Akinfe and Loyinmi 2021). The method of solution demonstrates the EPDTM's flexible efficiency when compared to other current classical approaches to solving the system of linear and nonlinear fractional differential equations. However, the hybrid scheme has not been used to solve other forms of differential equations. It is in this view that this study aims to solve the Rosenau–Hyman equation and the Korteweg–de Vries (KdV) equation using the proposed Elzaki Projected Differential Transform Method.

Materials and Methods

Rosenau–Hyman and the Korteweg–de Vries (KdV) equation

Rosenau–Hyman equation

The Rosenau-Hyman equation was utilized as a simplified model for the study of nonlinear dispersion in pattern generation in liquid droplets, and it has numerous applications in the modeling of various engineering and physics issues (Lawal et al. 2019b, Arslan 2020, Erinle-Ibrahim et al. 2020, Kumbinarasaiah and Adel 2021). The Rosenau–Hyman equation or $K(n,n)$ equation is a KdV-like equation having compaction solutions.

This nonlinear partial differential equation is of the form

$$u_t + a(u^n)_x + (u^n)_{xxx} = 0 \tag{1}$$

The equation is named after Phillip Rosenau and James M. Hyman, who used it in their 1993 study of compactions.

Korteweg–de Vries (KdV) equation

The Korteweg–de Vries (KdV) equation is a nonlinear, dispersive partial differential equation for a function of two real variables, space x and time t:

$$\partial_t \phi + \partial_x^3 \phi - 6\phi \partial_x \phi = 0$$

with ∂_x and ∂_t denoting partial derivatives with respect to *x* and *t*.

The constant 6 in front of the last term is conventional but of no great significance: multiplying *t*, *x*, and ϕ by constants can be used to make the coefficients of any of the three terms equal to any given non-zero constants (Miura et al. 1968).

Elzaki projected differential equation method

Considering a general non-linear homogenous partial differential equation with the initial condition:

$$Ru(x,t) + Mu(x,t) + Nu(x,t) = 0 \tag{2}$$

Subject to the initial condition

$$u(x,0) = g(x) \tag{3}$$

Where: $\boldsymbol{R}$ is a linear differential operator of order one, $\boldsymbol{M}$ is a linear differential operator of less order than $\boldsymbol{R}$, $\boldsymbol{N}$ is the general nonlinear differential operator, and 0 is the source term (Lu et al. 2018, Suleman et al. 2018).

Taking the Elzaki Transform on both sides of the equation, to get:

$$E[Ru(x,t)] + E[Mu(x,t)] + E[Nu(x,t)] = 0 \tag{4}$$

Using the differentiation property of Elzaki Transform and the above initial conditions, we have

$$\left[\frac{E[u(x,t)]}{v} - vu(x,0)\right] + E[Mu(x,t) + Nu(x,t)] = 0 \tag{5}$$

$$E[u(x,t)] = v^2 g(x) - vE[Mu(x,t) + Nu(x,t)] \tag{6}$$

Applying the inverse Elzaki transform on both sides of equation (6), we get:

$$u(x,t) = g(x) - E^{-1}\{vE[Mu(x,t) + Nu(x,t)]\} \tag{7}$$

Where $g(x)$ is the first of the series and the prescribed initial condition

$$\sum_{k=0}^{\infty} u(x,k+1) = -E^{-1}\{vE[Mu(x,t)+Nu(x,t)]\} \tag{8}$$

Now, we apply the projected differential transform method.

$u(x,k+1) = -E^{-1}\{vE[A_k + B_k]\}$ Where A_k and B_k are the projected differential transform of $Mu(x,t)$ and $Nu(x,t)$ respectively.

At k = 0, 1, 2, 3, …, n, …

$u(x,1) = -E^{-1}\{vE[A_0 + B_0]\}$, $u(x,2) = -E^{-1}\{vE[A_1 + B_1]\}$, $u(x,3) = -E^{-1}\{vE[A_2 + B_2]\}, \ldots,$ $u(x,n+1) = -E^{-1}\{vE[A_n + B_n]\}$

Then, the general approximate solution of the EPDTM is given by

$$u(x,t) = u(x,0) + u(x,1) + u(x,2) + u(x,3) + \ldots \tag{9}$$

and

$$u(x,t) = \sum_{k=0}^{\infty} u(x,k) \tag{10}$$

Applications

Case 1: Consider the Rosenau-HymanK (2, 2) equation

$$u_t + u_x^2 + u_{xxx}^2 = 0 \tag{11}$$

Subject to

$$u(x,0) = x \tag{12}$$

By taking the Elzaki transform of the equation

$$E[u_t] + E[u_x^2 + u_{xxx}^2] = 0 \tag{13}$$

$$\left[\frac{T(u,v)}{v} - vu(x,0)\right] + E[u_x^2 + u_{xxx}^2] = 0$$

$$T(u,v) - v^2 u(x,0) + vE[u_x^2 + u_{xxx}^2] = 0$$

$$T(u,v) = v^2 u(x,0) - vE[u_x^2 + u_{xxx}^2] \tag{14}$$

By applying the inverse Elzaki transform, we get

$$u(x,t) = u(x,0) - E^{-1}[vE(u_x^2 + u_{xxx}^2)] \tag{15}$$

$$u(x,t) = x - E^{-1}[vE(u_x^2 + u_{xxx}^2)] \tag{16}$$

$$\sum_{k=0}^{\infty} u(x,k+1) = -E^{-1}[vE(u_x^2 + u_{xxx}^2)] \tag{17}$$

Now, by applying the projected differential transform method, we get

$$u(x,k+1) = -E^{-1}[vE(A_k + B_k)] \text{ For } k = 0,1,2,3,\ldots.. \tag{18}$$

Where $A_k = \left[\sum_{n=0}^{k} u(x,n)u(x,k-n)\right]_x$ and $B_k = \left[\sum_{n=0}^{k} u(x,n)u(x,k-n)\right]_{xxx}$ are the projected differential transform of u_x^2 and u_{xxx}^2, respectively.

At k = 0,

$$u(x,1) = -E^{-1}[vE(A_0 + B_0)] \tag{19}$$

$A_o = [u(x,0)u(x,0)]_x = [(x)(x)]_x = [x^2]_x = 2x$;

$B_o = [u(x,0)u(x,0)]_{xxx} = [(x)(x)]_{xxx} = [x^2]_{xxx} = 0$

$u(x,1) = -E^{-1}[vE(2x)] = -E^{-1}[v \cdot v^2 \cdot (2x)] = -E^{-1}[v^3 \cdot (2x)] = -(2x)t$

$$u(x,1) = -2xt \tag{20}$$

At k = 1, $u(x,2) = -E^{-1}[vE(A_1 + B_1)]$

$A_1 = [u(x,0)u(x,1) + u(x,1)u(x,0)]_x = [(x)(-2xt) + (-2xt)(x)]_x = [-2x^2t - 2x^2t]_x = [-4x^2t]_x = -8xt$;

$B_1 = [u(x,0)u(x,1) + u(x,1)u(x,0)]_{xxx} = [(x)(-2xt) + (-2xt)(x)]_{xxx} = [-2x^2t - 2x^2t]_{xxx} = [-4x^2t]_{xxx} = 0$

$u(x,2) = -E^{-1}[vE(-8xt)] = -E^{-1}[v \cdot v^3 \cdot (-8x)] = -E^{-1}[v^4 \cdot (-8x)] = -\left[(-8x)\frac{t^2}{2!}\right]$

$$u(x,2) = 4xt^2 \tag{21}$$

At k = 2, $u(x,3) = -E^{-1}[vE(A_2 + B_2)]$

$A_2 = [u(x,0)u(x,2) + u(x,1)u(x,1) + u(x,2)u(x,0)]_x = [(x)(4xt^2) + (-2xt)(-2xt) + (4xt^2)(x)]_x =$;
$[4x^2t^2 + 4x^2t^2 + 4x^2t^2]_x = [12x^2t^2]_x = 24xt^2$

$B_2 = [u(x,0)u(x,2) + u(x,1)u(x,1) + u(x,2)u(x,0)]_{xxx} = [(x)(4xt^2) + (-2xt)(-2xt) + (4xt^2)(x)]_{xxx}$
$= [4x^2t^2 + 4x^2t^2 + 4x^2t^2]_{xxx} = [12x^2t^2]_{xxx} = 0$

$u(x,3) = -E^{-1}[vE(24xt^2)] = -E^{-1}[v \cdot 2!v^4 \cdot (24x)] = -E^{-1}[v^5 \cdot 48x] = -\left[48x \cdot \frac{t^3}{3!}\right]$

$$u(x,3) = -8xt^3 \tag{22}$$

At k = 3, $u(x,4) = -E^{-1}[vE(A_3 + B_3)]$

$A_3 = [u(x,0)u(x,3) + u(x,1)u(x,2) + u(x,2)u(x,1) + u(x,3)u(x,0)]_x =$
$[(x)(-8xt^3) + (-2xt)(4xt^2) + (4xt^2)(-2xt) + (-8xt^3)(x)]_x = [-8x^2t^3 - 8x^2t^3 - 8x^2t^3 - 8x^2t^3]_x$;
$= [-32x^2t^3]_x = -64xt^3$

$B_3 = [u(x,0)u(x,3) + u(x,1)u(x,2) + u(x,2)u(x,1) + u(x,3)u(x,0)]_{xxx}$
$= [(x)(-8xt^3) + (-2xt)(4xt^2) + (4xt^2)(-2xt) + (-8xt^3)(x)]_{xxx} = [-8x^2t^3 - 8x^2t^3 - 8x^2t^3 - 8x^2t^3]_{xxx}$
$= [-32x^2t^3]_{xxx} = 0$

$u(x,4) = -E^{-1}[vE(-64xt^3)] = -E^{-1}[v \cdot 3!v^5 \cdot (-64x)] = -E^{-1}[v^6 \cdot 3! \cdot -64x] = -\left[-64x \cdot 3! \cdot \frac{t^4}{4!}\right]$

$$u(x,4) = 16xt^4 \tag{23}$$

We obtain the respective solutions of the equation as:

$u(x,0) = x$; $u(x,1) = -2xt$; $u(x,2) = 4xt^2$; $u(x,3) = -8xt^3$; $u(x,4) = 16xt^4$

Then, the solution to the K (n, n) equation according to EPDTM is given as:

$$u(x,t) = \sum_{k=0}^{\infty} u(x,k) \tag{24}$$

$$u(x,t) = u(x,0) + u(x,1) + u(x,2) + u(x,3) + ...$$

$$u(x,t) = x - 2xt + 4xt^2 - 8xt^3 + 16xt^4 ....$$

$$u(x,t) = x(1 - 2t + 4t^2 - 8t^3 + 16t^4 ....) \tag{25}$$

Using a Mathematical computational package, the above multivariate series solution converges to the closed form which is a replica of the exact solution

$$u(x,t) = \frac{x}{1+2t} \tag{26}$$

Case 2: Consider the modified Korteweg-de-Vries equation

$$u_t - 6uu_x + u_{xxx} = 0 \tag{27}$$

Subject to

$$u(x,0) = 6x \tag{28}$$

By taking the Elzaki transform of the equation

$$E[u_t] - E[6uu_x - u_{xxx}] = 0 \tag{29}$$

$$\frac{T(u,v)}{v} - vu(x,0) - E(6uu_x - u_{xxx}) = T(u,v) - v^2 u(x,0) - vE(6uu_x - u_{xxx}) = 0$$

$$T(u,v) = v^2 u(x,0) + vE(6uu_x - u_{xxx}) \tag{30}$$

By applying the inverse Elzaki transform of the equation

$$u(x,t) = u(x,0) + E^{-1}[vE(6uu_x - u_{xxx})]$$

$$u(x,t) = 6x + E^{-1}[vE(6uu_x - u_{xxx})] \tag{31}$$

Where

$$\sum_{k=0}^{\infty} u(x,k+1) = E^{-1}[vE(6uu_x - u_{xxx})] \tag{32}$$

Now by applying the Projected differential transform method, where

$$u(x,k+1) = E^{-1}[vE(6A_k - B_k)] \text{ For } k = 0,1,2,3,.... \tag{33}$$

Where $A_k = \sum_{n=0}^{k} u(x,n)u(x,k-n)_x$ and $B_k = u(x,k)_{xxx}$ is the projected differential transform of uu_x and u_{xxx}

At k = 0,

$$u(x,1) = E^{-1}[vE(6A_0 - B_0)] \tag{34}$$

$A_0 = u(x,0)u(x,0)_x = (6x)(6x)_x = (6x)(6) = 6^2 x$; $B_0 = u(x,0)_{xxx} = (6x)_{xxx} = 0$

$$u(x,1) = E^{-1}[vE(6 \cdot 6^2 x - 0)] = E^{-1}[vE(6^3 x)] = E^{-1}[v^3 (6^3 x)]$$

$$u(x,1) = 6^3 xt \tag{35}$$

At k = 1,

$$u(x,2) = E^{-1}[vE(6A_1 - B_1)] \tag{36}$$

$A_1 = u(x,0)u(x,1)_x + u(x,1)u(x,0)_x = (6x)(6^3 xt)_x + (6^3 xt)(6x)_x = (6x)(6^3 t) + (6^3 xt)(6) = 2 \cdot 6^4 xt$;

$B_1 = u(x,1)_{xxx} = (6^3 xt)_{xxx} = 0$

$$u(x,2) = E^{-1}\left[vE(6\cdot 2\cdot 6^4 xt - 0)\right] = E^{-1}\left[vE(2\cdot 6^5 xt)\right] = E^{-1}\left[v^4(2\cdot 6^5 x)\right] = \frac{t^2}{2!}\cdot 2\cdot 6^5 x$$

$$u(x,2) = 6^5 xt^2 \tag{37}$$

At k = 2,

$$u(x,3) = E^{-1}\left[vE(6A_2 - B_2)\right] \tag{38}$$

$$A_2 = u(x,0)u(x,2)_x + u(x,1)u(x,1)_x + u(x,2)u(x,0)_x = (6x)(6^5 xt^2)_x + (6^3 xt)(6^3 xt)_x + (6^5 xt^2)(6x)_x;$$
$$= (6x)(6^5 t^2) + (6^3 xt)(6^3 t) + (6^5 xt^2)(6) = 3\cdot 6^6 xt^2$$

$$B_2 = u(x,2)_{xxx} = (6^5 xt^2)_{xxx} = 0$$

$$u(x,3) = E^{-1}\left[vE(6\cdot 3\cdot 6^6 xt^2 - 0)\right] = E^{-1}\left[vE(3\cdot 6^7 xt^2)\right] = E^{-1}\left[2!\cdot v^5\cdot 3\cdot 6^7 x\right] = \frac{t^3}{3!}\cdot 2!\cdot 3\cdot 6^7 x$$

$$u(x,3) = 6^7 xt^3 \tag{39}$$

At k = 3,

$$u(x,4) = E^{-1}\left[vE(6A_3 - B_3)\right] \tag{40}$$

$$A_3 = u(x,0)u(x,3)_x + u(x,1)u(x,2)_x + u(x,2)u(x,1)_x + u(x,3)u(x,0)_x = (6x)(6^7 xt^3)_x$$
$$+ (6^3 xt)(6^5 xt^2)_x + (6^5 xt^2)(6^3 xt)_x + (6^7 xt^3)(6x)_x = (6x)(6^7 t^3) + (6^3 xt)(6^5 t^2) + (6^5 xt^2)(6^3 t) +$$
$$(6^7 xt^3)(6) = 4\cdot 6^8 xt^3$$

$$B_3 = u(x,3)_{xxx} = (6^7 xt^3)_{xxx} = 0$$

$$u(x,4) = E^{-1}\left[vE(6\cdot 4\cdot 6^8 xt^3 - 0)\right] = E^{-1}\left[vE(4\cdot 6^9 xt^3)\right] = E^{-1}\left[3!\cdot v^6\cdot 4\cdot 6^9 x\right]$$
$$= \frac{t^4}{4!}\cdot 3!\cdot 4\cdot 6^9 x = 6^9 xt^4 \tag{41}$$

Then, the solution of the Korteweg-de Vries equation according to EPDTM is given as:

$$u(x,t) = \sum_{k=0}^{\infty} u(x,k) = 6x + 6^3 xt + 6^5 xt^2 + 6^7 xt^3 + 6^9 xt^4 \ldots\ldots$$

$$u(x,t) = 6x(1 + 6^2 t + 6^4 t^2 + 6^6 t^3 + 6^8 t^4 \ldots\ldots) \tag{42}$$

Using a Mathematical computational package, the above multivariate series solution converges to the closed form which is a replica of the exact solution

$$u(x,t) = \frac{6x}{1 - 36t} \tag{43}$$

Case 3: Consider the Modified Korteweg-de Vries equation

$$u_t - 6uu_x - u_{xxx} = 0 \tag{44}$$

With the initial condition

$$u(x,0) = (1 - x) \tag{45}$$

By taking the Elzaki transform of the equation

$$E[u_t] - 6E[uu_x] - E[u_{xxx}] = 0 \tag{46}$$

$$\left[\frac{T(u,v)}{v} - vu(x,0)\right] - 6E[uu_x] - E[u_{xxx}] = \left[\frac{T(u,v)}{v} - vu(x,0)\right] - E[6uu_x + u_{xxx}] = 0$$

$$T(u,v) - v^2 u(x,0) - vE[6uu_x + u_{xxx}] = 0 \tag{47}$$

By applying the inverse Elzaki transform, we get

$$u(x,t)-u(x,0)-E^{-1}\left[vE\left[6uu_x+u_{xxx}\right]\right]=0$$

$$u(x,t)=u(x,0)+E^{-1}\left[vE\left[6uu_x+u_{xxx}\right]\right]=(1-x)+E^{-1}\left[vE\left[6uu_x+u_{xxx}\right]\right]$$

$$u(x,t)=\sum_{k=0}^{\infty}u(x,k) \tag{48}$$

Now, by applying the projected differential transform method

$$\therefore u(x,k+1)=E^{-1}\left[vE\left[6A_k+B_k\right]\right] \tag{49}$$

Where $A_k=\sum_{n}^{k}u(x,n)u(x,k-n)_x$ and $B_k=u(x,k)_{xxx}$ is the projected differential transform of the nonlinear part of the equation uu_x and u_{xxx}, respectively.

At k = 0,

$$u(x,1)=E^{-1}\left[vE\left[6A_0+B_0\right]\right] \tag{50}$$

$$A_0=u(x,0)u(x,0)_x=(1-x)(1-x)_x=(1-x)(-1)=x-1;\ B_0=(1-x)_{xxx}=0$$

$$u(x,1)=E^{-1}\left[vE\left[6(x-1)\right]\right]=E^{-1}\left[v^3\left[6(x-1)\right]\right]$$

$$u(x,1)=-6(1-x)t \tag{51}$$

At k = 1,

$$u(x,2)=E^{-1}\left[vE\left[6A_1+B_1\right]\right] \tag{52}$$

$$A_1=u(x,0)u(x,1)_x+u(x,1)u(x,0)_x=(1-x)\left[-6(1-x)t\right]_x+\left[-6(1-x)t\right](1-x)_x$$
$$=(1-x)(6t)+\left[-6(1-x)t\right](-1)=6t-6xt+6t-6xt=2\cdot 6(1-x)t\ ;$$
$$B_1=-\left[6(1-x)t\right]_{xxx}=0$$

$$u(x,2)=E^{-1}\left[vE\left[2\cdot 6^2(1-x)t\right]\right]=E^{-1}\left[v^4\left[2\cdot 6^2(1-x)\right]\right]=2\cdot 6^2(1-x)\frac{t^2}{2!}$$

$$u(x,2)=6^2(1-x)t^2 \tag{53}$$

At k = 2,

$$u(x,3)=E^{-1}\left[vE\left[6A_2+B_2\right]\right] \tag{54}$$

$$A_2=u(x,0)u(x,2)_x+u(x,1)u(x,1)_x+u(x,2)u(x,0)_x=(1-x)\left[6^2(1-x)t^2\right]_x+\left[6(1-x)t\right]\left[6(1-x)t\right]_x+\left[6^2(1-x)t^2\right](1-x)_x$$
$$=(1-x)(-6^2t^2)+\left[6(1-x)t\right](-6t)+\left[6^2(1-x)t^2\right](-1)=-6^2t^2+6^2xt^2-6^2t^2+6^2xt^2-6^2t^2+6^2xt^2=-3\cdot 6^2(1-x)t^2$$
$$;\ B_2=\left[6^2(1-x)t^2\right]_{xxx}=0$$

$$u(x,3)=E^{-1}\left[vE\left[-3\cdot 6^3(1-x)t^2\right]\right]=E^{-1}\left[2!v^5\left[-3\cdot 6^3(1-x)\right]\right]=-3!\cdot 6^3(1-x)\frac{t^3}{3!}$$

$$u(x,3)=-6^3(1-x)t^3 \tag{55}$$

At k = 3,

$$u(x,4)=E^{-1}\left[vE\left[6A_3+B_3\right]\right] \tag{56}$$

$$A_3=u(x,0)u(x,3)_x+u(x,1)u(x,2)_x+u(x,2)u(x,1)_x+u(x,3)u(x,0)_x=(1-x)\left[-6^3(1-x)t^3\right]_x+\left[-6(1-x)t\right]\left[6^2(1-x)t^2\right]_x$$
$$+\left[6^2(1-x)t^2\right]\left[-6(1-x)t\right]_x+\left[-6^3(1-x)t^3\right](1-x)_x=(1-x)(6^3t^3)\left[-6(1-x)t\right](-6^2t^2)+\left[6^2(1-x)t^2\right](-6t)+\left[6^3(1-x)t^3\right](-1)$$
$$=6^3t^3-6^3xt^3+6^3t^3-6^3xt^3+6^3t^3-6^3xt^3+6^3t^3-6^3xt^3=4\cdot 6^3(1-x)t^3$$
$$;\ B_3=\left[-6^3(1-x)t^3\right]_{xxx}=0$$

$$u(x,4) = E^{-1}\left[vE\left[4\cdot 6^4(1-x)t^3\right]\right]$$
$$u(x,4) = E^{-1}\left[3!v^6\left[4\cdot 6^4(1-x)\right]\right]$$
$$u(x,4) = 4!\cdot 6^4(1-x)\frac{t^4}{4!}$$
$$u(x,4) = 6^4(1-x)t^4 \qquad (57)$$

Then, the solution to the Korteweg-de Vries equation according to EPDTM is given as:

$$u(x,t) = \sum_{k=0}^{\infty} u(x,k) \ = u(x,0) + u(x,1) + u(x,2) + u(x,3) + \dots$$
$$u(x,t) = (1-x) - 6(1-x)t + 6^2(1-x)t^2 - 6^3(1-x)t^3 + 6^4(1-x)t^4 \dots$$
$$u(x,t) = (1-x)\left[1 - 6t + (6t)^2 - (6t)^3 + (6t)^4 \dots\right] \qquad (58)$$

Using a Mathematical computational package, the above multivariate series solution converges to the closed form which is a replica of the exact solution

$$u(x,t) = \frac{1-x}{1+6t} \qquad (59)$$

Results and Discussion

Results

In this section, we checked for the efficacy, convergence, and authenticity of the proposed Elzaki Projected differential transform method (EPDTM) in providing an approximate and reliable solution to the Rosenau-Hyman, Korteweg-de-Vries, and Korteweg-de-Vries-Burgers equations by comparing results with the exact solution. The exact results are easily obtained by the Taylor's series.

Case 1: The Rosenau Hyman [K (n, n)] equation

To validate the efficacy of the method, we have presented

Table **1** which compares the exact results and our proposed EPDTM results.

Table 1: Exact and asymptomatic results of the Rosenau Hymans K (2, 2) equation. With parameter x = 0.1, 0.2, 0.3, and 0.5, for each value of t = 0.01, 0.02, 0.03, 0.04 and 0.05

Case 1	t	Exact	EPDTM	Error = |Exact – EPDTM
	0.001	0.009980039920	0.009980039920	0
	0.002	0.009960159363	0.009960159363	0
	0.003	0.009940357853	0.009940357853	0
x = 0.01	0.004	0.009920634921	0.009920634921	0
	0.005	0.009900990099	0.009900990100	0.00000000001
	0.001	0.01996007984	0.01996007984	0
	0.002	0.01992031873	0.01992031873	0
	0.003	0.01988071571	0.01988071571	0
x = 0.02	0.004	0.01984126984	0.01984126984	0
	0.005	0.01980198020	0.01980198020	0
	0.001	0.02994011976	0.02994011976	0
	0.002	0.02988047809	0.02988047809	0
x = 0.03	0.003	0.02982107356	0.02982107356	0

Case 1	t	Exact	EPDTM	Error = |Exact – EPDTM
	0.004	0.02976190476	0.02976190476	0
	0.005	0.02970297030	0.02970297030	0
x = 0.04	0.001	0.03992015968	0.03992015968	0
	0.002	0.03984063745	0.03984063745	0
	0.003	0.03976143141	0.03976143141	0
	0.004	0.03968253968	0.03968253968	0
	0.005	0.03960396040	0.03960396040	0
x = 0.05	0.001	0.04990019960	0.04990019960	0
	0.002	0.04980079681	0.04980079682	0.0000000001
	0.003	0.04970178926	0.04970178926	0
	0.004	0.04960317460	0.04960317460	0
	0.005	0.04950495050	0.04950495050	0

Case 2: The Korteweg-De-Vries equation

To validate the efficacy of the method, we have presented Table 2 which compares the exact results and our proposed EPDTM results.

Table 2: Exact and asymptomatic results of the modified Korteweg de Vries equation. With parameter x = 0.01, 0.02, 0.03, 0.04 and 0.05, for each value of t = 0.001, 0.002, 0.003, 0.004 and 0.005

Case 2	**t**	**Exact solution**	**EPDTM solution**	**Error = | Exact – EPDTM|**
x = 0.01	0.001	0.06224066388	0.06224066390	0.0000000001
	0.002	0.06465517242	0.06465517241	0.0000000001
	0.003	0.06726457398	0.06726457397	0.0000000001
	0.004	0.07009345794	0.07009345768	0.0000000026
	0.005	0.07317073170	0.07317072909	0.0000000261
x = 0.03	0.001	0.1867219917	0.1867219917	0
	0.002	0.1939655173	0.1939655172	0.0000000001
	0.003	0.2017937220	0.2017937219	0.0000000001
	0.004	0.2102803738	0.2102803730	0.0000000008
	0.005	0.2195121951	0.2195121873	0.0000000078
x = 0.04	0.001	0.2489626556	0.2489626555	0.0000000001
	0.002	0.2586206897	0.2586206896	0.0000000001
	0.003	0.2690582960	0.2690582958	0.0000000002
	0.004	0.2803738318	0.2803738308	0.0000000010
	0.005	0.2926829268	0.2926829163	0.0000000005
x = 0.05	0.001	0.3112033195	0.3112033195	0
	0.002	0.3232758620	0.3232758621	0.0000000001
	0.003	0.3363228700	0.3363228700	0
	0.004	0.3504672897	0.3504672884	0.0000000013
	0.005	0.3658536486	0.3658536455	0.0000000031

Case 3: Modified Korteweg de Vries equation

To validate the efficacy of the method, we have presented Table 3 which compares the exact results and our proposed EPDTM results.

Table 3: Exact and asymptomatic results of the modified Korteweg de Vries equation. With parameter x = 0.1, 0.2, 0.3, and 0.5, for each value of t = 0.01, 0.02, 0.03, 0.04 and 0.05

Case 3	t	Exact solution	EPDTM solution	Error = |Exact – EPDTM|
	0.001	0.9840954274	0.9840954274	0
	0.002	0.9782608696	0.9782608698	0.0000000002
x = 0.01	0.003	0.9724950884	0.9724950903	0.0000000019
	0.004	0.9667968750	0.9667968827	0.0000000077
	0.005	0.9611650485	0.9611650719	0.0000000234
	0.001	0.9741550696	0.9741550696	0
x = 0.02	0.002	0.9683794466	0.9683794468	0.0000000002
	0.003	0.9626719057	0.9626719075	0.0000000018
	0.004	0.9570312500	0.9570312576	0.0000000076
	0.005	0.9514563107	0.9514563338	0.0000000231
	0.001	0.9642147117	0.9642147117	0
	0.002	0.9584980237	0.9584980239	0.0000000002
x = 0.03	0.003	0.9528487230	0.9528487230	0.0000000018
	0.004	0.9472656250	0.9472656326	0.0000000076
	0.005	0.9417475728	0.9417475957	0.0000000229
	0.001	0.9542743539	0.9542743539	0
	0.002	0.9486166008	0.9486166010	0.0000000002
x = 0.04	0.003	0.9430255403	0.9430255421	0.0000000018
	0.004	0.9375000000	0.9375000075	0.0000000075
	0.005	0.9320388350	0.9320388576	0.0000000226
	0.001	0.9443339960	0.9443339960	0
	0.002	0.9387351779	0.9387351781	0.0000000002
x = 0.05	0.003	0.9332023576	0.9332023594	0.0000000018
	0.004	0.9277343750	0.9277343824	0.0000000074
	0.005	0.9223300971	0.9223301195	0.0000000224

Solution and convergence plots of the Exact and EPDTM solutions

The solution plots for the cases 1–3 are presented in Figures 1–6. Also, convergence analysis plots are showed in Figures 7–9.

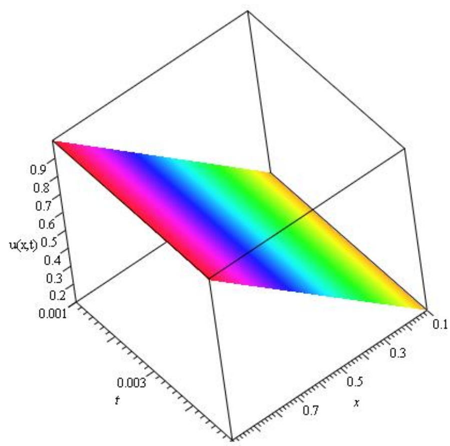

Figure 1: Solution plot of the exact solution of the Rosenau Hyman [K (n, n)] equation (Case 1).

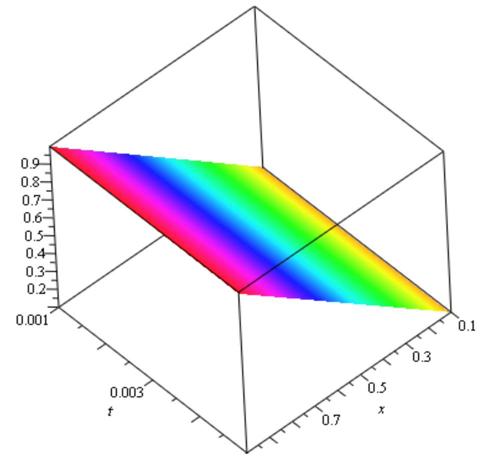

Figure 2: Solution plot of the EPDTM solution of the Rosenau Hyman [K (n, n)] equation (Case 1).

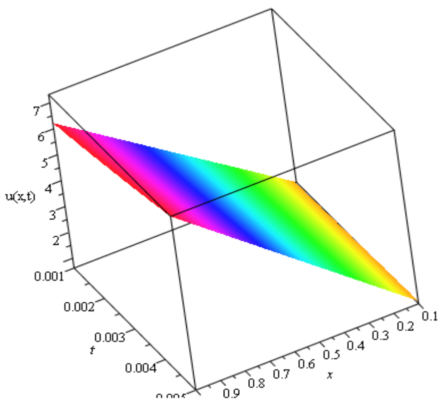

Figure 3: Solution plot of the exact solution of the Korteweg-De-Vries equation (Case 2).

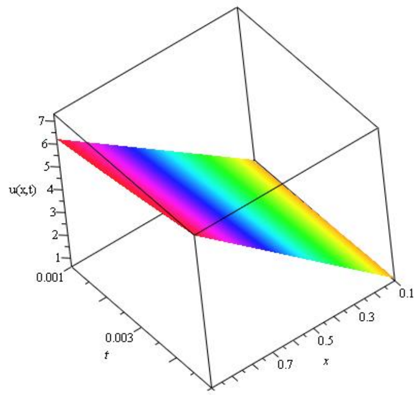

Figure 4: Solution plot of the EPDTM solution of the Korteweg-De-Vries equation (Case 2).

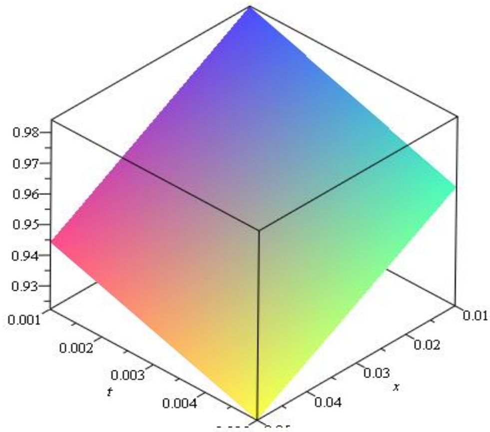

Figure 5: Solution plot of the exact solution of the Modified Korteweg-De-Vries equation (Case 3).

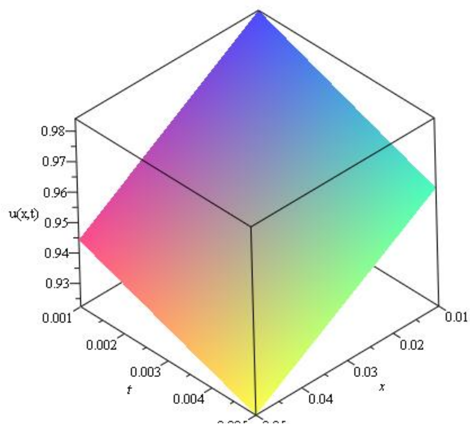

Figure 6: Solution plot of the EPDTM solution of the Modified Korteweg-De-Vries equation (Case 3).

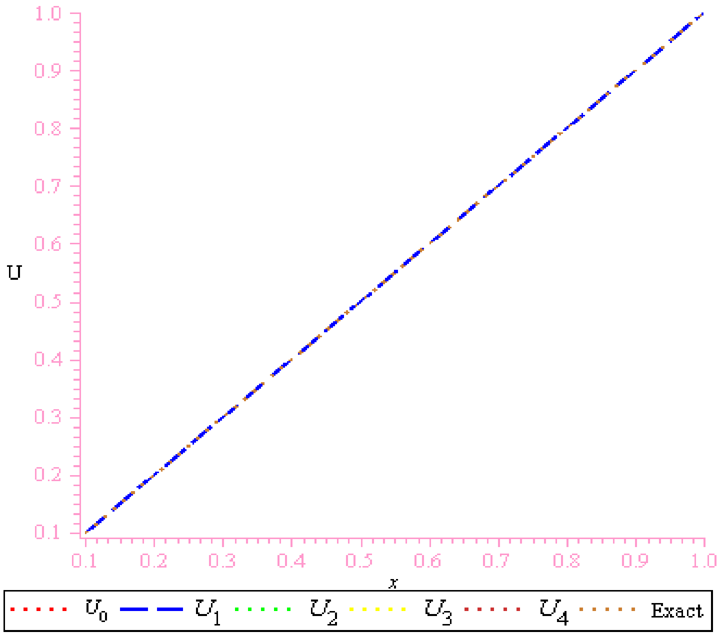

Figure 7: Convergent plot of the EPDTM solution (Case 1) at t = 0.001.

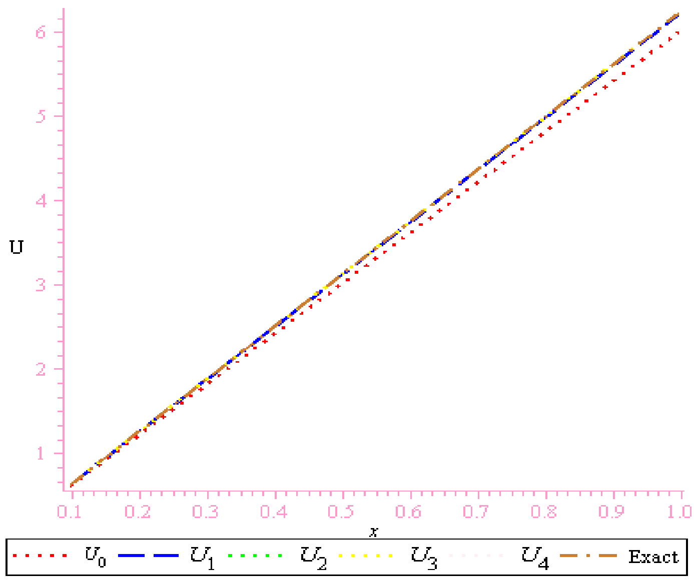

Figure 8: Convergent plot of the EPDTM solution (Case 2) at t = 0.001.

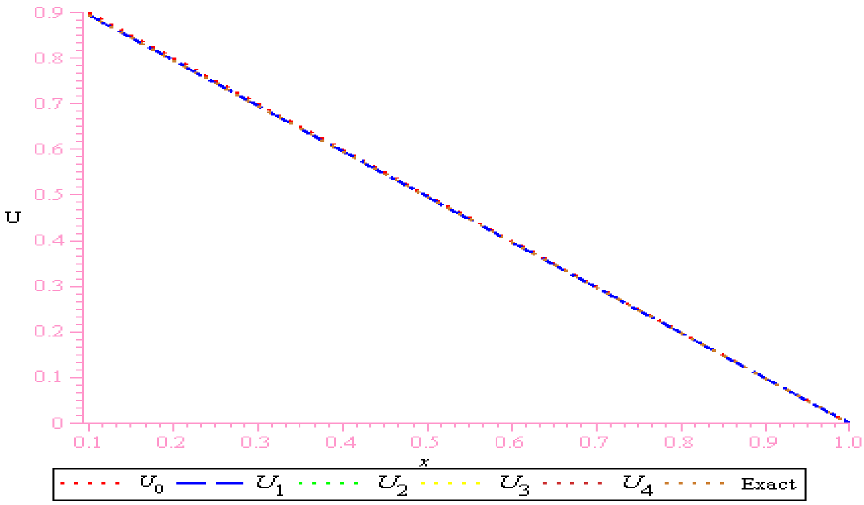

Figure 9: Convergent plot of the EPDTM solution (Case 3) at t = 0.001.

Discussion of Results

In the research work, an efficient hybrid method has been utilized which involves the coupling of the Elzaki transform and Projected differential transform method in finding the approximate solution to the Rosenau Hyman [K (n, n)] equation and Korteweg-de-Vries equation (KdV). Elzaki Projected differential transform method has been implemented excellently on the K (n, n) and KdV equations; thereby obtaining a solution that is highly convergent and accurate. With the merging of techniques, the Elzaki transform takes care of the linear terms in the linear terms in equations and an asymptotic technique Projected differential transform method to treat the nonlinear terms in the equation makes the convergence of the results obtained faster and highly accurate (see Tables 1–3). Also, Figure 1,Figure 2,Figure 3,Figure 4,Figure 5, and Figure 6 show the solution plots for the three cases at t = 0.001. The comparisons consist of the exact results extracted from prominent literature that have implanted the normal analytical means and Elzaki Projected differential transform (EPDTM) results. Figure 7, Figure 8 and Figure 9 show the convergence plot for the EPDTM. The results validated the efficacy and reliability of the EPDTM by showing high levels of convergence results.

Conclusion

The results obtained using the Elzaki Projected differential method showed that the method is valid, reliable, and highly efficient in solving non-linear partial differential. When we compared the results to the exact solution via tables and graphs, the convergence and stability of the method were ascertained. As a result of the fast convergence and efficiency of the Elzaki Projected differential method (EPDTM), we hereby recommend this method (EPDTM) in obtaining an approximate solution which exact solution can also be determined from the multivariate series.

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