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Mathematical Characterization of Biological Control of Cassava Pests Model

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Abstract

Pests are major constraints to the effective growth and development of every crop through their damage, and can be controlled effectively by the use of their natural enemies which is referred to as the biological pest control. In this study, the biological control model of cassava pests through optimal control theory was presented in order to minimize the population of the pests and stabilize the natural enemies population so as not to affect the crop negatively. A mathematical model was formulated via the Lotka-Volterra model, and the model was characterized. The optimality system was established, the equilibrium point with its uniqueness was established for the model. Finally, stability analysis of the model was investigated through optimal control approach and numerical data were employed to validate the system. The results obtained showed that cassava pests can be effectively controlled biologically.

Keywords: Optimal control, Cassava pest, Biological control, Stability, Natural enemies.

Introduction

Optimal control is simply defined as the best control or the best way to control a dynamic system over a period of time to minimize a performamce index, (Burghes and Graham 1982, Aida-zade and Ragimov 2007, Derouich et al. 2014). Any living organism that is injurious or causes loss or irritation to other organisms (plants or animals) is known as a pest. Most farmers employ the use of chemicals such as insecticides which have side effects on plants, humans and the environment at large if not carefully used in controlling pests (Anguelov et al. 2017).

The biological control involves the introduction of the natural enemies of pests to control or keep the pests population under control. In other words, it is the control which involves the use of parasitoids, predators and pathogens to maintain the population of pests at a level lower than it would without natural enemies (Andres et al. 1979, Rafikov and Balthazar 2005, Rafikov et al. 2008, Michael 2008, Lucas 2011). According to FAO (2020) Cassava (*Manihot esculenta* Graitz) is the fourth most important source of calories in Africa and as a result quick measures should be taken to control its pests.

Many researchers have worked on cassava modelling and application of optimal control to pests population. Moreno-Cadena et al. (2021) worked on the modelling of growth and development of cassava. Chapwanya and Dumont (2021) investigated the interactions among crops, vector and virus with cassava mosaic virus disease. Alemneh et al. (2021) studied the optimal deterministic eco-epidemiological model for the dynamics of virus disease. Anguelov et al. (2017) determined the genric model to study the impact of mating disruption control to reduce the population of the pests. Picart et al. (2011) worked on optimal control problem of insect pest populations. Rafikov and Balthazzar (2005) and Rafikov et al. (2008) studied optimal control of pests in population dynamics by obtaining the pest control strategy through natural enemies introduction also known as prey-predator relationship between the soybean caterpillar *(Anticarsia gemmatalis)* and its predators *(Nabis spp, Geocoris* arachnid, etc.*)* using the Lotka-Volterra model. Zhou (2015) presented a predator-prey system with seasonal reproduction by proposing various theoretical and empirical approaches to develop mathematical and statistical tools for studying predator- prey interactions. Dabbs (2010) considered optimal control in form of decreasing the growth rate of the pest population using discrete time models. Mahapatra and Santra (2016) worked on prey-predator model for optimal harvesting with functional response, incorporating prey refuges. Krishna (2011) studied predator-prey dynamics with disease in predator. Aderinto et al. (2013) worked on a qualitative study of biological pest control system. Herren and Neuenschwander (1991) worked on biological control of cassava pests. Gutierrez et al. (1988) worked on analysis of biological control of cassava pests. Lucas (2011) also analyzed predator-prey model using system of ordinary linear differential equations where he focused on the predation relationship between two specific species in the central regions of Canada, the Canadian Lynx and the Snowshore Hare. The Lotka-Volterra model was formulated for analyzing the differential equations; it was found that the population of the two species fluctuates on an average of ten year periodic oscillations.

However, in this paper, concentration is on the optimal biological control of two major pests of cassava known as the green spider mite *(Mononychellus tanajoa)* and cassava mealybug *(Phenacoccus manihoti)* using their predators known as the predatory mites botanically called *Typhlodromalus aripo* and *Epidinocarsis lopezi*, respectively using the method of predator-prey model called Lotka-Volterra in an attempt to maximize cassava outputs by farmers by minimizing the population of pests below injury level and stabilize the predator within the level appropriate to pest control.

The green spider mite is a greenish/yellowish pest which is recognized mostly by its inconspicuous body segmentation. It originated from the neotropical area, but was first discovered in Africa in the early 1970s. Over 60 countries have records of *Mononychellus tanajoa* which makes it a major and widespread pest of cassava and has spread throughout the cassava belt of Africa. Cassava mealybug *(Phenacoccus manihoti)* being indigenous to South Africa has spread in Africa to practically all countries where cassava is grown. This pest is recognized by its body segmentation which bears very short, lateral and caudal white filaments in the forms of swellings that produce a toothed appearance to the body outline; the eyes are relatively prominent, legs also are well developed and of equal size (Matile-Ferrero 1977, Gutierrez et al. 1988, Mustiya et al. 2014, Moreno-Cadena et al. 2021). Because of the importance of cassava in Africa as a staple food, quick measures should be taken to control these pests. Different measures of control can be used, but none is as promising and effective as the biological control. Good cultural practices such as intercropping, and manipulation of planting time are not sufficient to control green spider mites. Chemical control also has its own side effects which make it unsuitable, hence the use of biological control.

Materials and Methods

Model descriptions and formulation

Let the population of the prey species be denoted by $M_1(t)$ and $M_2(t)$, and the population of the predator species be denoted by $T_1(t)$ and $T_2(t)$. Since the prey population grows in the absence of predator, we have

$$\frac{dM_1}{dt} = aM_1 \qquad (1)$$

and $\frac{dM_2}{dt} = bM_2$ (2)

for the two species, respectively; where a, and b are positive (i.e. $a > 0$, $b > 0$) and $T_1 = 0$ and $T_2 = 0$. The predator dies out in the absence of the prey, hence

$$\frac{dT_1}{dt} = -cT_1 \quad (3)$$

and

$$\frac{dT_2}{dt} = -dT_2 \quad (4)$$

where $c, d > 0$, $M_1, M_2 = 0$.

When the prey and predator interact, their encounter is proportional to the product of their population and tends to promote the growth of the predator and impede the growth of the prey. Thus, the predator population increases by βMT and the prey population decrease by αMT with respect to each of the species.

The model is given as

$$\begin{aligned}
\frac{dM_1}{dt} &= aM_1 - \alpha_1 M_1 T_1 - bM_2 \\
\frac{dM_2}{dt} &= bM_2 - \alpha_2 M_2 T_2 - aM_1 \\
\frac{dT_1}{dt} &= -cT_1 + \beta_1 M_1 T_1 - \beta_2 T_2 - u_1 a \\
\frac{dT_2}{dt} &= -dT_2 + \beta_2 M_2 T_2 - \beta_1 T_1 - u_2 b
\end{aligned} \quad (5)$$

$a, b, c, d > 0$.

With the assumptions that the prey has unlimited food supply, the predator depends completely on its prey as the only source of food, and each prey has no other threats except for its predator under consideration. The flow diagram for the model is presented in Figures 1-3, and the definitions of the parameters are given in Table 1.

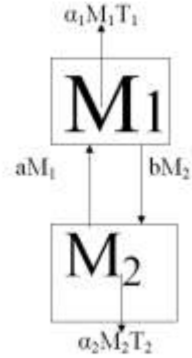

Figure 1: Flow diagram for the model.

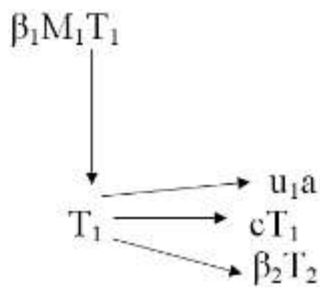

Figure 2: Flow diagram for the model.

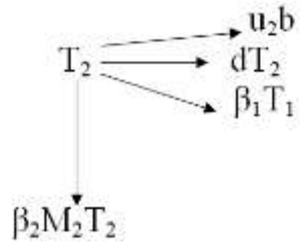

Figure 3: Flow diagram for the model.

Table 1 : Definition of parameters of the model

Parameter	Definitions
M_1	Population of prey species 1
M_2	Population of prey species 2
T_1	Population of predator species 1
T_2	Population of predator species 2
a	Birth rate of prey species 1
b	Birth rate of prey species 2
c	Death rate of predator species 1
d	Death rate of predator species 2
α_1	Death rate per encounter of prey species 1 due to predation
α_2	Death rate per encounter of prey species 2 due to predation
β_1	Growth rate of predator species 1
β_2	Growth rate of predator species 2
u_1	Control for species 1
u_2	Control for species 2

Results and Discussion

Optimality system

The objective function for the optimal control problem is given by

Minimize

$$J(u) = \int_{t_0}^{t_f} \left[cM_i + Au_i^2 \right] dt, \quad (6)$$

$$i = 1, 2, \ 0 \le u_i \le 1$$

where A is the cost associated with using biological control, u is the biological control effort, c is the appropriate constant associated with the number of preys. We choose u such that $0 \le u \le 1$ for effective introduction of predators at a given rate. The optimal control problem stated as;

Minimize

$$J(u) = \int_{t_0}^{t_f} \left[cM_i + Au_i^2 \right] dt \qquad i = 1,2 \qquad 0 \le u_i \le 1$$

subject to the constraints

$$\frac{dM_1}{dt} = aM_1 - \alpha_1 M_1 T_1 - bM_2 \qquad (7)$$

$$\frac{dM_2}{dt} = bM_2 - \alpha_2 M_2 T_2 - aM_1$$

$$\frac{dT_1}{dt} = -cT_1 + \beta_1 M_1 T_1 - \beta_2 T_2 - u_1 a$$

$$\frac{dT_2}{dt} = -dT_2 + \beta_2 M_2 T_2 - \beta_1 T_1 - u_2 b$$

with $M_i(0) = M_0 \qquad T_i(0) = T_0$ for $i = 1,2$

Uniqueness of the equilibrium of the system

Assumption 1: If f_i and $g_i(i=1,2,\cdots,n)$ are Lipschitz continuous, then $\exists$ r, s such that

$$\left|f_i(u)-f_i(v)\right|\le r_i|u-v|,|g_i(u)-g_i(v)|\le s_i|u-v| \qquad (8)$$

for all $u,v\in R$ and $i=1,2,\cdots,n$ (Barbu 1994).

Theorem 1

Given that the assumption (1) is satisfied, then equation (7) has a unique equilibrium point.

Proof:

Let $Q_1=(M_i,J_i)^T$ and $Q_2=(T_i,k_i)^T$ denote the two equilibrium points of the model $i=1,2$,

where $M_i=(M_1,M_2)^T, J_i=(J_1,J_2)^T, T_i=(T_1,T_2)^T, K_i=(K_1,K_2)^T$

Then,

$$\begin{aligned}
&a_iM_{1i}-\alpha_{1i}M_{1i}T_{1i}-b_iM_{2i}=0\\
&b_iM_{2i}-\alpha_{2i}M_{2i}T_{2i}-a_iM_{1i}=0\\
&a_iJ_{1i}-\alpha_{1i}J_{1i}T_{1i}-b_iJ_{2i}=0\\
&b_iJ_{2i}-\alpha_{2i}J_{2i}T_{2i}-a_iJ_{1i}=0
\end{aligned} \qquad (9)$$

and

$$\begin{aligned}
&-c_iT_{1i}+\beta_{1i}M_{1i}T_{1i}-\beta_{2i}T_{2i}-u_{1i}a_i=0\\
&-d_iT_{2i}+\beta_{2i}M_{2i}T_{2i}-\beta_{1i}T_{1i}-u_{2i}b_i=0\\
&-c_iK_{1i}+\beta_{1i}M_{1i}K_{1i}-\beta_{2i}K_{2i}-u_{1i}a_i=0\\
&-d_iK_{2i}+\beta_{2i}M_{2i}T_{2i}-\beta_{1i}T_{1i}-u_{2i}b_i=0
\end{aligned} \qquad (10)$$

This implies that

$$\begin{aligned}
&a_i(M_{1i}-J_{1i})=\alpha_{1i}J_{1i}(M_{1i}-J_{1i})+b_i(M_{2i}-J_{2i})\\
&b_i(M_{2i}-J_{2i})=\alpha_{2i}J_{2i}(M_{2i}-J_{2i})+a_i(M_{1i}-J_{1i})
\end{aligned} \qquad (11)$$

and

$$\begin{aligned}
&\beta_{1i}M_{1i}(T_{1i}-K_{1i})=c_i(T_{1i}-K_{1i})+\beta_{2i}(T_{2i}-K_{2i})\\
&\beta_{2i}M_{2i}(T_{2i}-K_{2i})=d_i(T_{2i}-K_{2i})+\beta_{1i}(T_{1i}-K_{1i})
\end{aligned} \qquad (12)$$

using (8), (11) becomes

$$\begin{aligned}
&a_i|M_{1i}-J_{1i}|\le r_1|\alpha_{1i}T_{1i}||M_{1i}-J_{1i}|+r_2b_i|M_{2i}-J_{2i}|\\
&b_i|M_{2i}-J_{2i}|\le r_1|\alpha_{2i}T_{2i}||M_{2i}-J_{2i}|+r_2a_i|M_{1i}-J_{1i}|
\end{aligned} \qquad (13)$$

Which implies

$$\begin{aligned}
&((a_i-r_1|\alpha_{1i}T_{1i}|)-r_2b_i)(|M_{1i}-J_{1i}|,|M_{2i}-J_{2i}|)\\
&((b_i-r_1|\alpha_{2i}T_{2i}|)-r_2a_i)(|M_{2i}-J_{2i}|,|M_{1i}-J_{1i}|)
\end{aligned} \qquad (14)$$

Multiplying both sides by $((a_i-r_1|\alpha_{1i}T_{1i}|)-r_2b_i)^{-1}$ and $((b_i-r_1|\alpha_{2i}T_{2i}|)-r_2a_i)^{-1}$ to obtain

$$(|M_{1i}-J_{1i}|,|M_{2i}-J_{2i}|)^T\le(0,0)^T$$

That is

$$|M_{1i} - J_{1i}| = 0 \Rightarrow M_{1i} = J_{1i} \qquad (15)$$

and

$$|M_{2i} - J_{2i}| = 0 \Rightarrow M_{2i} = J_{2i} \qquad (16)$$

Also by using (8), (12) becomes

$$\begin{aligned} &\beta_{1i} M_{1i}(T_{1i} - K_{1i}) \le s_1|c_i||T_{1i} - K_{1i}| + s_2|\beta_{2i}||(T_{2i} - K_{2i})| \\ &\beta_{2i} M_{2i}(T_{2i} - K_{2i}) \le s_1|d_i||T_{2i} - K_{2i}| + s_2|\beta_{1i}||(T_{1i} - K_{1i})| \end{aligned} \qquad (17)$$

We then have

$$\begin{aligned} &((\beta_{1i} M_{1i} - s_1|c_i|) - s_2|\beta_{2i}|)(|T_{1i} - K_{1i}|, |T_{2i} - K_{2i}|)^T \le (0,0)^T \\ &((\beta_{2i} M_{2i} - s_1|d_i|) - s_2|\beta_{1i}|)(|T_{2i} - K_{2i}|, |T_{1i} - K_{1i}|)^T \le (0,0)^T \end{aligned} \qquad (18)$$

Multiplying both sides by $((\beta_{1i} M_{1i} - s_1|c_i|) - s_2|\beta_{2i}|)^{-1}$ and $((\beta_{2i} M_{2i} - s_1|d_i|) - s_2|\beta_{1i}|)^{-1}$, respectively to obtain

$$(|T_{1i} - K_{1i}|, |T_{2i} - K_{2i}|)^T \le (0,0)^T$$

which implies that

$$|T_{1i} - K_{1i}| = 0 \quad \Rightarrow T_{1i} = K_{1i} \qquad (19)$$

and

$$|T_{2i} - K_{2i}| = 0 \quad \Rightarrow T_{2i} = K_{2i} \qquad (20)$$

This proves the uniqueness of the equilibrium point of the model.

Stability analysis of the system

Theorem 2: Given equation (5), the equilibrium point of the system is stable.

Proof: Considering (7), to obtain the equilibrium point we set the derivatives to zero

$$\begin{aligned} 0 &= aM_1 - \alpha_1 M_1 T_1 - bM_2 \\ 0 &= bM_2 - \alpha_2 M_2 T_2 - aM_1 \\ 0 &= -cT_1 + \beta_1 M_1 T_1 - \beta_2 T_2 - u_1 a \\ 0 &= -dT_2 + \beta_2 M_2 T_2 - \beta_1 T_1 - u_2 b \end{aligned} \qquad (21)$$

The nature of the model near the equilibrium point is determined by linearization of the system. The Jacobian matrix J is given as

$$\mathbf{J(M_1, M_2, T_1, T_2)} = \begin{pmatrix} a - \alpha_1 T_1 & -b & -\alpha_1 M_1 & 0 \\ -a & b - \alpha_2 T_2 & 0 & -\alpha_2 M_2 \\ \beta_1 T_1 & 0 & -c + \beta_1 M_1 & -\beta_2 \\ 0 & \beta_2 T_2 & -\beta_1 & -d + \beta_2 M_2 \end{pmatrix} \qquad (22)$$

The equilibrium point is given by $(M_1^*,\ M_2^*,\ T_1^*,\ T_2^*) = (0,\ 0,\ u_1a,\ u_2b)$.

$$= \begin{pmatrix} a-\alpha_1 u_1 a & -b & 0 & 0 \\ -a & b-\alpha_2 u_2 b & 0 & 0 \\ \beta_1 u_1 a & 0 & -c & -\beta_2 \\ 0 & \beta_2 u_2 b & -\beta_1 & -d \end{pmatrix} \qquad (23)$$

The stability of the equilibrium point is established from the roots of the corresponding eigenvalue equation, $det(A-\lambda I) = 0$. The following parameter values were used:

a = 0.17, b = 0.116, $\alpha_1 = \alpha_2 = 0.20$, c = d = 0.00017, $\beta_1 = \beta_2 = 0.0085$. Aderinto et al. (2013)

$$0 \le u_1 \le 1, \quad 0 \le u_2 \le 1$$

$$\mathbf{A} = \begin{pmatrix} 0.153 & -0.116 & 0 & 0 \\ -0.17 & 0.102 & 0 & 0 \\ 0.00072 & 0 & -0.00017 & -0.0085 \\ 0 & 0.00059 & -0.0085 & -0.00017 \end{pmatrix} \qquad (24)$$

$$det(A-\lambda I) = \begin{vmatrix} 0.153-\lambda & -0.116 & 0 & 0 \\ -0.17 & 0.102-\lambda & 0 & 0 \\ 0.00072 & 0 & -0.00017-\lambda & -0.0085 \\ 0 & 0.00059 & -0.0085 & -0.00017-\lambda \end{vmatrix} = 0 \qquad (25)$$

The eigenvalues was computed using Maple 18, the characteristic equation obtained as

$$= 2.97117605E-7 + 0.0000170176205\lambda - 0.0042729211\lambda^2 - 0.25466\lambda^3 + \lambda^4 \qquad (26)$$

and the eigenvalues were obtained as:

$$\lambda_1 = 0.008329999996, \lambda_2 = 0.2702243847, \lambda_3 = -0.008669999987, \lambda_4 = -0.01522438475$$

Discussion of results

The results obtained showed that the equilibrium point for the parameter values of the system can be stable, despite the fact that not all the eigenvalues are negative but less than 1. In other words, cassava pest population can be minimized below damaged level and natural enemies population can be steadilised within the level appropriate to pest control.

Conclusion

In this study, a mathematical model of the biological control of two major pests of cassava was presented. The optimality system was established, the system equilibrium was found to be unique and stable. Numerical values were employed to test for the validity of the model, and the results obtained showed that cassava pests can be effectively controlled biologically without the use of chemicals, and the system can be stable.

Conflict of interests: Authors declare no conflict of interest regarding this work.

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