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On Scalar Multiples of Hypercyclic Operators on Non-normable and Separable Fréchet Spaces

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Abstract

In this paper, we proved that if F is a non-normable and separable Fréchet space without a continuous norm, then there exists an operator $T \in L(F)$ such that λT is hypercyclic for any $\lambda \in \mathbb{C} \setminus \{0\}$ of modulus 1 and has similar set of hypercyclic vectors as T. An illustrative example to the main theorem is also provided.

Keywords: Non-normable Fréchet space, Hypercyclic operator, Supercyclic operator.

Introduction

Density of an orbit (or hypercyclicity) of continuous linear operators on non-normable Fréchet spaces has been considered by several authors like Gethner and Shapiro (1987), Godefroy and Shapiro (1991) and many others.

Rolewicz (1969) proved that no linear operator on a finite-dimensional Banach space is hypercyclic. He raised the question in 1969: whether or not every separable infinite-dimensional Banach space carries a hypercyclic continuous linear operator. This was recently and independently answered in a positive sense by Ansari (1997) and Bernal-González (1999). This result was extended to complete and metrizable locally convex spaces (Fréchet spaces) by Bonet and Peris (1998). Gethner and Shapiro (1987) and Kitai (1984) independently found an adequate condition for hypercyclicity that produces the results of Rolewicz (1969), Birkhoff (1929), and MacLane (1952) and gave new cases of operators with hypercyclic vectors.

In 1991, Godefroy and Shapiro studied the hypercyclicity of partial differential operators with constant coefficients on Fréchet spaces with a continuous norm as $H(\mathbb{C}^N)$, the space of all entire functions on $\mathbb{C}^N$ or without a continuous norm as $C^{\infty}(\mathbb{R}^N)$, the space of infinitely continuous real valued functions on $\mathbb{R}^N$.

Bonet (2010) in his short expository article stated an open problem about the linear structure of non-normable Fréchet spaces and related this question with some other open problems about continuous linear operators on Fréchet spaces. Bonet (2010) stated the following problem: Is there a non-normable Fréchet space F such that every continuous linear operator T on F has the form $T = \lambda I + S$, where S maps a 0-neighbourhood of F into a bounded set? For example, Bonet (2010) related this problem with the following question:

Question 1.1 *(Bonet 2010): Does every non-normable separable infinite dimensional Fréchet space admit a hypercyclic operator T such that λT is hypercyclic for all $\lambda \neq 0$? The most important case is for infinite dimensional Fréchet Montel spaces.*

Below are the notions about the

development of hypercyclicity:

Definition 1.1 (Bonet and Peris 1998): An operator T on a locally convex space F is called *hypercyclic* if $Orb\,(T, v) := \{v, Tv, T^2v, \cdots\}$ is dense in F for some $v \in F$, that is, $\overline{Orb(T,v)} = \overline{\{v, Tv, T^2v, \ldots\}} = F$.
In this case, v is a hypercyclic vector for T.

Definition 1.2 (Bonet and Peris 1998): An operator $T \in L\,(F)$ is said to be *supercyclic* if there is a vector $y \in F$ such that the set $\{\lambda T^k y : \lambda \in \mathbb{C},\ k = 0, 1, 2, \cdots\}$ is dense in F. The vector y with this property is called supercyclic vector for T.

In this work, we present the scalar multiples of hypercyclic operators on non-normable and separable Fréchet spaces. We respond to Question 1.2 below due to Frerick and Peris (2012) by first considering the existing partial solutions to the question and then develop our results. Throughout this paper, an operator means a continuous linear map and $L\,(F)$ is the space of all operators T: $F \to F$.

The main goal of this paper is to investigate the following question:

Question 1.2: *If F is a non-normable, separable Fréchet space, is there an operator $T \in L\ (F)$ such that $\lambda\ T$ is hypercyclic for any $\lambda \in \mathbb{C}\backslash\{0\}$?*

Preliminaries

To establish the main results for this paper, we will require the following definitions, propositions, lemmas, theorems and corollaries:

Definition 2.1 (Bayart and Matheron 2009): Let F be a topological vector space. The operator $T \in L(F)$ is said to satisfy the hypercyclicity criterion if there exists an increasing sequence of integers (n_k), two dense sets $A, B \subset F$ and a sequence of maps $S_{n_k}: B \to F$ such that

(i) $T^{n_k}(x) \to 0$ for any $x \in A$

(ii) $S_{n_k}(y) \to 0$ for any $y \in B$

(iii) $T^{n_k} S_{n_k}(y) \to y$ for each $y \in B$.

Definition 2.2 (Bonet and Peris 1998): A continuous linear operator T on a locally convex space F is called *Montel* if it maps each bounded set into compact subsets of F, while T is compact if it maps a 0-neighborhood in F into a compact subset of F.

Remark 2.1 (Bonet and Peris 1998): Each bounded operator on a Fréchet Montel space is compact and each compact operator is Montel, and the opposite holds for Banach spaces yet not in general, as the identity on an infinite-dimensional Fréchet Montel space appears.

Definition 2.3: A Fréchet space (or complete metrizable locally convex space) is said to be *non-normable* if it is not isomorphic to a normed space, or in the same meaning if it has no bounded 0-neighborhood.

Bonet and Peris (1998) proved that, every non-normable, separable infinite dimensional Fréchet space supports a continuous hypercyclic operator. The following results were established:

Theorem 2.1 *(Bonet and Peris 1998): Every separable infinite dimensional Fréchet space admits a hypercyclic surjective operator.*

Lemma 2.1 *(Bonet and Peris 1998): Let F be a separable infinite dimensional Fréchet space, $F \neq \omega$. There are sequences $(x_k)_k \subset F$ and $(f_k)_k \subset F'$ such that*

(i) $(x_k)_k$ converges to 0 in F, and span$\{x_k ; k \in \mathbb{N}\}$ is dense in F.

(ii) $(f_k)_k$ is F-equi-continuous in F'.

(iii) $\langle x_k, f_m\rangle = 0$ *if* $k \neq m$ *and* $\left(\langle x_k, f_k\rangle\right)_k \subset (0,1)$.

Lemma 2.2 *(Bonet and Peris 1998): Let F be a locally convex space. Let $(x_k)_k \subset F$ and*

$(f_m)_m \subset F'$ *satisfy the following conditions:*

(a) $(x_k)_k$ *converges to* $0 \in F$ *, and the closed absolutely convex cover C of* $(x_k)_k$ *satisfies that* F_c *is a Banach space and* F_c *is dense in F, where* F_c *is a locally convex space F sub the closed absolutely convex cover C of* $(x_k)_k$.

(b) $(f_m)_m$ *is F-equi-continuous in* F'

(c) $\langle x_k, f_m\rangle = 0$ *if* $k \neq m$ *and* $(\langle x_k, f_m\rangle)_k$ *is a bounded sequence in* $(0, \infty)$.

Then the operator $T: F \to F$,

$Tx := x + \sum_{K=1}^{\infty} 2^{-k}\langle x, f_{k+1}\rangle x_k$ *where* $x \in F$ *, is hypercyclic, open, has closed range (hence is surjective) and has finite dimensional kernel.*

The following result was shown by Bonet (2010) for a non-normable Fréchet space:

If F_0 is a non-normable Fréchet space such that each operator $T \in L(F_0)$ is of the form $T = \lambda I + S$ with S bounded, then no hypercyclic operator T on F_0 satisfies that μT is hypercyclic for all $\mu \neq 0$; hence solving question 1.2 described above in the negative sense.

Proposition 2.1. (Bonet 2010) If S is a bounded operator on a Fréchet space F and $\lambda \in \mathbb{C}$, then there is $\mu > 0$ such that $\mu(\lambda I + S)$ is not hypercyclic on F.

León-Saavedra and Müller (2004) proved that, let T be a bounded linear operator acting on a separable complex Banach space Y. Let $T \in B(Y)$ be a hypercyclic operator and λ a complex number of modulus 1, then λT is hypercyclic and has a similar set of hypercyclic vectors as T.

An adaptation of this outcome allows characterizations for a wide class of supercyclic operators: Let $T \in B(Y)$. The vector $v \in Y$ is said to be supercyclic for T if and only if the set $\{rT^k v: r > 0, k = 0, 1, \cdots\}$ is dense in Y. This offers answers to several questions examined in the literature.

In appropriate way, to solve Question 1.2 it is sufficient to demonstrate that a non-normable and separable Fréchet space F admits a continuous linear operator T such that λT is hypercyclic for every $\lambda > 0$.

Frerick and Peris (2012) established the following result:

Proposition 2.2 (Frerick and Peris 2012): Every separable Fréchet space F without a continuous norm supports a continuous linear operator $T \in L(F)$ such that λT is hypercyclic for every $\lambda \neq 0$.

We intend to show that in fact λT is hypercyclic for every non-zero $\lambda \in \mathbb{C}$ of modulus 1. Additionally, T and λT have similar sets of hypercyclic vectors if $\lambda = e^{2\pi ir}$ where r is a rational number. The main result also has a significant outcome in the supercyclicity setting.

The concept of $\mathbb{R}^+$-supercyclicity is defined below.

Definition 2.4 (León-Saavedra and Müller 2004): A vector z is said to be $\mathbb{R}^+$-supercyclic for the operator T if the set $\{t T^k z: t > 0, k = 0, 1, 2, \cdots\}$ is dense.

Herrero (1991) discovered that, there are two kinds of supercyclic operators T, which are: Operators fulfilling $\sigma_p(T^*) = \emptyset$ where σ_p denotes the point spectrum and operators with $\sigma_p(T^*) = \{\alpha\}$ for some non-zero $\alpha \in \mathbb{C}$; (here we have dim ker$(T^* - \alpha)$ = 1 = dim ker$(T^* - \alpha)^n$ for every $n \geq 1$).

The following theorem and corollary were proved by León-Saavedra and Müller (2004):

Theorem 2.2 *(León-Saavedra and Müller 2004): Let* $N \subset B(Y)$ *be a semigroup of operators and let* $y \in Y$ *satisfy that the set* $\{\lambda Sy: S \in N, \lambda \in \mathbb{C}, |\lambda| = 1\}$ *is dense in Y. Assume that there exists an operator* $T \in B(Y)$ *with* $\sigma_p(T^*) = \emptyset$ *satisfying* $TS = ST$ *for each* $S \in N$*. Then the set* $\{Sy: S \in N\}$ *is dense.*

Corollary 2.1 (León-Saavedra and Müller 2004): *Let* $T \in B(Y)$*. Then* $y \in Y$ *is hypercyclic for T if and only if the set* $\{\lambda T^k y: \lambda \in \mathbb{T}$ *(where* $\mathbb{T}$ *is a unit circle),* $k = 0, 1,$

$2, \cdots\}$ is dense in Y.
The following corollary follows from Theorem 2.2 above:

Corollary 2.2 *(León-Saavedra and Müller 2004): Let $T \in B(Y)$ be hypercyclic and $\lambda \in \overline{T}$. Then the operator λT is hypercyclic and has similar set of hypercyclic vectors as T.*

Main Results
Before we state our main results by responding to the Proposition 2.2, we need to deduce several observations from Theorem 2.1 above, as follows:

Observation 3.1
It is notable that, the countable product of lines ω: $= K^{\mathbb{N}}$ (where "K" is a scalar field) produced with the product topology is a separable Fréchet space which supports a hypercyclic operator. For example, the backward shift

$T: \omega \to \omega$,
$T(x_1, x_2, x_3, x_4 \ldots) := (x_2, x_3, x_4, x_5, \ldots)$
is hypercyclic.
The verification is prompt by the hypercyclicity criterion. It does the trick to discover dense subsets A and B of

$\omega(A = B := \bigoplus_{k\in\mathbb{N}} K)$
where "K" is a scalar field and a map
$S : B \longrightarrow \omega, \quad S(y_1, y_2, y_3, y_4, \cdots):$
$= \quad (0, y_1, y_2, y_3, \cdots)$

such that $(T^k x)_k$ converges pointwise to 0 for all $x \in A$, $TSy = y$ and $(S^k y)_k$ converges to 0 for every $y \in B$. Likewise we need to demonstrate the outcome for a separable infinite dimensional Fréchet space $F \neq \omega$.

Observation 3.2
We notice the following results:

Lemma 3.1: *Let F be a separable infinite dimensional Fréchet space, $F \neq \omega$.There are sequences $(x_k)_k \subset F$ and $(f_k)_k \subset F'$, such that*

(i) $(x_k)_k$ converges to 0 in F, and $span\{x_k; k \in \mathbb{N}\}$ is dense in F.
(ii) $(f_k)_k$ is F-equi-continuous in F'.
(iii) $\langle x_k, f_m\rangle = 0$ if $k \neq m$ and $(\langle x_k, f_k\rangle)_k \subset (0,1)$.

Proof. From the result by Metafune and Moscatelli (1989), there exists a dense subspace D of F which has a continuous norm p. Let us choose a linearly independent sequence $(y_k)_k \in D$ whose linear span is dense in D and therefore in F.

Through a classical method of Klee applied to the dual pair $(D, (D, p))$ (as used by Carreras and Bonet (1987), we discover sequences $(z_k)_k \subset D$, $(u_m)_m \subset (D, p)$ to such an extent that
span $\{z_k; k \in \mathbb{N}\}$ = span$\{y_k; k \in \mathbb{N}\}$,
and $z_k, u_m = \delta_{k,m}$ where $k, m \in \mathbb{N}$.
$\forall m \; \exists \; k_m \geq 1$ such that
$|u_m(x)| \leq K_m p(x), \; \forall x \in D$.
There exists a continuous seminorm q on F whose restrictions to D agrees with p. Every u_m has a unique continuous extension to F denoted again by u_m since D is dense in F.
Indeed,
$\forall m, |u_m(x)| \leq K_m \; q(x), \; \forall x \in F$.
Consequently, $\{K_m^{-1} u_m; \; m \in \mathbb{N}\}$ is equicontinuous on F'. There exists a sequence $(\alpha_k)_k \subset (0, 1)$ such that $x_k := \alpha_k z_k$; $k \in \mathbb{N}$ converges to $0 \in F$ since F is metrizable. Therefore, by setting
$f_m := K_m^{-1} u_m$ where $m \in \mathbb{N}$
we get (i) → (ii) → (iii).

Lemma 3.2: *Let F be a locally convex space. Let $(x_k)_k \subset F$ and $(f_m)_m \subset F'$ satisfy the following conditions:*
(a) $(x_k)_k$ converges to $0 \in F$, and the closed absolutely convex cover C of $(x_k)_k$ fulfils that F_c is a Banach space and F_c is dense in F.
(b) $(f_m)_m$ is F-equi-continuous in F'

(c) $\langle x_k, f_m\rangle = 0$ *if* $k \neq m$ *and* $(\langle x_k, f_m\rangle)_k$ *is a bounded sequence in* $(0, \infty)$.

Then the operator $T: F \to F$,

$Tx := x + \sum_{K=1}^{\infty} 2^{-k}\langle x, f_{k+1}\rangle x_k$ *where* $x \in F$, *is hypercyclic, open, has closed range (hence is surjective) and has finite dimensional kernel.*

Proof. From Carreras and Bonet (1987), the closed absolutely convex cover C of $(x_k)_k$ is compact and agrees with

$$\left\{\sum_{k=1}^{\infty} \alpha_k x_k : \sum_{k=1}^{\infty} |\alpha_k| \leq 1\right\}$$

Indeed $T := id_F + S$, with

$$Sx := \sum_{K=1}^{\infty} 2^{-k}\langle x, f_{k+1}\rangle x_k$$

where $x \in F$,

By equicontinuity of $(f_k)_k$, it means that $S := F \to F$ maps a 0-neighbourhood into a relatively compact set. From Grothendieck (1973), Theorem 1), T is open onto the range (that is homomorphism) with closed range and finite dimensional kernel.

Let us define

$Q : l_1 \to F_C$, $\quad Q((\alpha_j)_j) := \sum \alpha_j x_j$,

which is surjective, linear and continuous. Through a result of Salas (1995), the operator

$\tilde{T} : l_1 \to l_1$, $\tilde{T}((\alpha_j)_j) :=$

$$\left(\alpha, + \frac{f_2(x_2)}{2}\alpha_2, a_2 + \frac{f_3(x_3)}{2^2}\alpha_3, \cdots\right)$$

is hypercyclic.

Additionally, $TQ = Q\tilde{T}$ on l_1. If $e \in l_1$ is a hypercyclic vector of $\tilde{T}$ on l_1 then the set

$$\{Q\tilde{T}^k e; k \in \mathbb{N}\} = \{T^k Qe; k \in \mathbb{N}\},$$

is dense in F_c. We conclude that, Qe is a hypercyclic vector of $T \in F$ since F_c is dense in F. Accordingly, the range of T (which is dense and closed) is F and T is surjective.

Remark 3.1: In fact, Lemma 3.2 enhances and simplifies Theorem 1 of Ansari (1997).

Observation 3.3

Since F is complete, the sequences $(x_k)_k \subset F$ and $(f_m)_m \subset F^{\prime}$ constructed in Lemma 3.1 fulfills all the assumptions of Lemma 3.2. Therefore, the operator T: $F \to F$ is hypercyclic and surjective.

We have shown that the operator T: $F \to F$ is hypercyclic and surjective. Now, we will show that for every non-normable and separable Fréchet space F without a continuous norm as $C^\infty(\mathbb{R}^N)$ the space of infinitely continuous real-valued functions on $\mathbb{R}^N$, λT is also hypercyclic for every non-zero $\lambda \in \mathbb{C}$ of modulus 1.

We now extend Theorem 2.2 and Corollary 2.1 from separable complex Banach spaces to non-normable and separable Fréchet spaces without a continuous norm. Analogous to Theorem 2.2 above, we present the following result:

Theorem 3.1: *Let* $N \subset L(F)$ *be a semigroup of operators and let* $y \in F$ *satisfy that the set* $\{\lambda Sy$: $S \in N, \lambda \in \mathbb{C}, |\lambda| = 1\}$ *is dense in* F. *Assume that there exists an operator* $T \in L(F)$ *with* $\sigma_p(T^*) = \phi$ *satisfying* $TS = ST$ *for each* $S \in N$. *Then the set* $\{Sy: S \in N\}$ *is dense.*

Proof. For every $v \in F$, we set $N_v = \overline{\{Sv : S \in N\}}$. For $v, w \in F$ we set $G_{v,w} = \{\lambda \in \mathbb{C} : |\lambda| = 1, \lambda w \in N_v\}$. Indeed, $G_{v,w}$ is a closed subset of the unit circle $\mathrm{T} = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$. We let F_0 be the set of all vectors $v \in F$ such that $\overline{\{\lambda Sv : S \in N, \lambda \in \mathrm{T}\}} = F$ and proceed with the following steps:

Step 3.1: *We let* $v \in F_0$. *At that point* $G_{v,w} \neq \emptyset$ $\forall$ $w \in F$.

There exists sequences $S_k \subset N$ and $\lambda_k \subset \mathrm{T}$ such that $\lambda_k S_k v \to w$, since the set $\overline{\{\lambda Sv : S \in N, \lambda \in \mathrm{T}\}}$ is dense in F. Moving toward a subsequence if necessary, it is possible to assume that λ_k is convergent, $\lambda_k \to \lambda$ for some $\lambda \in \overline{T}$.

Then,

$$\|S_k v - \lambda^{-1} w\| \leq \|S_k v - \lambda_k^{-1}\| + \|(\lambda_k^{-1} - \lambda^{-1})\, w\| \longrightarrow 0.$$

Therefore, $\lambda^{-1} \in G_{v,w}$.

Step 3.2: *We let* $v, w, u \in F, \lambda_1 \in G_{v,w}$ *then* $\lambda_1\lambda_2 \in G_{vw}$ *and* $\lambda_2 \in G_{w,U}$. *Then* $\lambda_1\lambda_2 \in G_{v,u}$.

We let $\varepsilon > 0$. Then、there exist $s_1 \in N$ such that $\|S_{1w} - \lambda_2 u\| < \frac{\varepsilon}{2}$ and $s_2 \in N$ such that $\|S_2 v - \lambda_1 w\| < \frac{\varepsilon}{2\|S_1\|}$.

Then

$$\|S_1 S_{2^v} - \lambda_1 \lambda_{2^u}\| \leq \|S_1(S_{2^v} - \lambda_{1^w})\| + \|\lambda_1(s_{1^w} - \lambda_{2^u})\| < \varepsilon$$

Hence $\lambda_1\lambda_2 \in G_{v,u}$.

Let us now fix $y \in F_0$. $G_{y,y}$ is a non-empty closed subsemigroup of the unit circle $\overline{\mathcal{T}}$ by step 3.1 and step 3.2 above. First, we assume $G_{y,y} = \overline{\mathcal{T}}$. Then step 3.1 and step 3.2 above infer that $G_{y,x} = \overline{\mathcal{T}}\, \forall y \in F$. In this way $N_y = F$, thus the set $\{S_y : S \in N\}$ is dense in F.

In the following step we suppose $G_{y,y} \neq \overline{\mathcal{T}}$ and prove that this supposition leads to the contradiction.

Step 3.3: *There exists* $n \in \mathbb{N}$ *such that* $G_{y,y} = \{e^{2\pi ij/n} : j = 0,1,\ldots,n-1\}$.

We let $r = \{t > 0 : e^{2\pi it} \in G_{y,y}\}$. Certainly, r > 0 since otherwise $G_{y,y}$ would be dense in T. Therefore, $e^{2\pi ir} \in G_{y,y}$.

Let $n = min\{k \in \mathbb{N} : kr \geq 1\}$. If $nr > 1$ then $e^{2\pi i(nr-1)} \in G_{y,y}\ an\,d\ 0 < n_r - 1 < r$ which is a contradiction with the definition of r.

Thus nr = 1 and

$$G_{y,y} \subset \{e^{2\pi ij/n} : j = 0, 1, \ldots, n-1\}$$

If there exists a

$\lambda \in G_{y,y} \setminus \{e^{2\pi ij/n} : J = 0,1,\ldots,n-1\}$.

Then,

$\lambda = e^{2\pi_i t} and\ j_0/n < t < (j_0 + 1)/n$ for some $j_0, 0 \leq j_0 \leq n-1$.

Then,

$\lambda \cdot e^{-2\pi i j_0/n} = e^{2^{\pi i(t - j_0/n)}} \in G_{y,y}$

where $0 < t - j_o/n < 1/n = r$

which is again the contradiction with the definition of r.

Therefore,

$\{e^{2\pi ij/n} : J = 0,1,\cdots,n-1\} = G_{y,y}$.

Step 3.4: *We let* $x \in F_0$. *Then there exists* $\lambda_x \in T$ *such that* $G_{y,x} = \{\lambda_x e^{2\pi ij/n} : j = 0,1\ldots,n-1\}$.

By step 3.1, there are $\lambda_x \in G_{x,y}\ and\ \alpha \in G_{x,y}$ and by step 3.2, we have $\lambda_x G_{y,y} \subset G_{y,x}\ and \propto G_{y,x} \in G_{y,y}$.

Specifically, the number of elements in $G_{y,x}$ is equal to the number of elements in $G_{y,y}$ and $G_{y,x} = \lambda_x G_{y,y} = \{\lambda_x e^{2\pi ij/n} : j = 0,1 \ldots n-1\}$.

Step 3.5: $(T - z)_y \in F_0, \forall z \in \mathbb{C}$.

We have

$$\overline{(T - z)F} = F\ since\ \sigma_P(T^*) = \emptyset$$

Therefore,

$$(T - z)\overline{\{\lambda S y : S \in N, \lambda \in T\}} \subset \overline{\{\lambda(T - z)y : S \in N, \lambda \in T\}} = (T - z)F,$$

which is a dense subset of *F*.

For every non-zero vector x in the subspace produced by *y* and *Ty*, we define

f(x) = λ^n where $\lambda \in G_{y,x}$.

Indeed, a function f *is* well defined by step 3.4 above.

Step 3.6: *f is a continuous function.*

We assume on the contrary that there exits non-zero vectors $v_k, v \in \vee\{y, Ty\}$ such that $v_k \to v\ and\ f(v_k) \nrightarrow f(v)$.

Without loss of generality, we suppose that the sequence $(f(v_k))$ converges to some $\alpha \in \overline{\mathrm{T}}$, $\alpha \neq f(v)$.

Letting $\lambda_k \in G_{y,v_k}$, we can suppose that $\lambda_k \to \lambda$ for some $\lambda \in \overline{\mathrm{T}}$. Then $\lambda_k v_k \in Ny\ and\ \lambda_k v \to \lambda v$. We now have $\lambda v \in Ny\ and\ \lambda \in G_{y,v}$ since N_y is closed.

Thus, $\alpha = lim\, f(v_k) = lim\, \lambda_k^n = \lambda^n = f(v)$, which is a contradiction. Hence *f* is continuous on the set $\vee\{y, Ty\} \setminus \{0\}$.

We now finish the proof of theorem 3.1 as follows:

Step 3.7: Let us start by showing that vectors y and Ty are linearly independent. Assume on the contrary that $Ty = \alpha y$

for some $\alpha \in \mathbb{C}$. Then
$S\, ker(T-\alpha) \subset ker(T-\alpha), \forall \in N$
and
$F = \overline{\{\lambda Sy: S \in N, \lambda \in \mathbb{T}\}} \subset ker(T-\alpha)$.

Therefore, T is a scalar multiple of the identity, which contradicts to the assumption that $\sigma_p(T^*) = \emptyset$.
Now let $D = \{z \in \mathbb{C}: |z| \leq 1\}$ indicate the unit disc and the function $g: D \rightarrow \overline{\mathcal{T}}$ be defined by

$$g(z) = f\big(zy + (1-|z|)T_y\big).$$

Indeed, g is a continuous function. Then
$G_{y,zy} = z^{-1}G_{y,y}$ *and* $g(z) = f(zy) = z^{-n}f(y) = z^{-n}$, $\forall z$ fulfilling $|z| = 1$.
By Rudin (1987) the function g does not exist, but rather gives a homotopy between a constant path $\varphi_1: \langle 0,2\pi\rangle \rightarrow \overline{\mathcal{T}}$ given by $\varphi_{1(t)} = g(0)$ and the path
$\varphi_2: \langle 0,2\pi\rangle \rightarrow \overline{\mathcal{T}}$
defined by $\varphi_2(t) = g(e^{it}) = e^{-nit}$, where *-n* is a winding number. Therefore, $G_{y,y} = \overline{\mathcal{T}}$ and the set
$\{Sy: S \in N\}$ is dense in F.

Analogous to corollary 2.1 above, we prevent the following result:

Corollary 3.1: Let $T \in L(F)$. Then $y \in F$ is hypercyclic *for T if and only if the set*
$\{\lambda T^k y: \lambda \in \mathbb{T},\ k = 0,1,2,\cdots\}$
is dense in F.
Proof. (⇒) Forward implication:
This is trivial.
(⇐) Backward implication:
We let $y \in F$ fulfil

$$\overline{\{\lambda T^k y: \lambda \in \mathbb{T}, k = 0,1,2,\ldots\}} = F$$

Letting N = $\{T^k: k = 0,1,2\ldots\}$, it is enough to prove that $\sigma_P(T^*) = \emptyset$.
Assume in other way that $\alpha \in \sigma_p(T^*)$ and let $y^* \in F^*$ be the relating eigenvector, $T^*y^* = \alpha y^*$. We have

$$\begin{aligned}\mathbb{C} &= \overline{\{\langle \lambda T^k y, y^*\rangle: \lambda \in \mathbb{T}, k = 0,1,2,\ldots\}} \\ &= \overline{\{\langle \lambda y, \alpha^k y^*\rangle: \lambda \in \mathbb{T}, k = 0,1,2,\ldots\}} \\ &= \langle y, y^*\rangle \cdot \overline{\{\lambda \alpha^k: \lambda \in \mathbb{T}, k = 0,1,2,\cdots 1\}}.\end{aligned}$$

If $|\alpha| > 1$ *and* $\langle y, y^*\rangle \neq 0$ then the set

$$\langle y, y^*\rangle \cdot \overline{\{\lambda \alpha^k: \lambda \in \mathbb{T}, k = 0,1,2,\cdots\}}$$

is bounded below and thus is not dense in $\mathbb{C}$, either.
Hence, $\sigma_P(T^*) = \emptyset$.
We now present an illustrative example to support the result above. We use Definition 2.1 and apply the results by Rolewicz (1969), and Bonet and Peris (1998) to construct this example, as follows:

Example 3.1: Let $T: \omega \rightarrow \omega$, defined by $T(x_0, x_1 \cdots) = (x_1, x_2 \cdots)$ be the hypercyclic backward shift operator on a separable infinite-dimensional Fréchet space $\omega := K^{\mathbb{N}}$ (that is, countably infinite product of the real or complex scalar field K) given the product topology. Then the operator λT is hypercyclic for any scalar λ such that $|\lambda| > 1$ and has similar sets of hypercyclic vector as *T.*
Proof. To verify the hypercyclic of λT, we apply the hypercyclicity criterion to the whole sequence $(n_k) := (k)$. There exists dense subsets A and B of ω such that $A = B := \bigoplus_{k\in\mathbb{N}} K$, where "K" is a scalar field and the maps $S_k: B \rightarrow \omega$ given by $S_k := \lambda^{-k}S^k$, where S is the forward shift operator defined by
$S(y_0, y_1, \cdots) = (0, y_0, y_1, \cdots)$
such that $((\lambda T)^k x)_k$ converges pointwise to 0 for all $x \in A$, $(S^k y)_k$ converges to 0 for every $y \in B$ and $(\lambda T)Sy = y, \forall y \in B$.

To see that the requirements (i), (ii) and (iii) of Definition 2.1 are satisfied, it is enough to note the following:
(a) holds because $(\lambda T)^k(x) \rightarrow 0, \forall x \in A$.
(b) holds because by applying norm to the map given by $S_k := \lambda^{-k}S^k$ we have

$$\begin{aligned}\|S_k\| &= \|\lambda^{-k}S^k\| \\ &\leq |\lambda|^{-k}\|S^k\| \\ \|S_k\| &\leq |\lambda|^{-k},\end{aligned}$$

since $\|S^k\| = 1$ *and* (c) holds because $(\lambda T)S = I_B$.

Conclusion

In this paper, we have proved that if F is a non-normable and separable Fréchet space without a continuous norm, then there exists an operator $T \in L$ *(F)* such that λ T is hypercyclic for any $\lambda \in \mathbb{C}\setminus\{0\}$ of modulus 1 and has similar set of hypercyclic vectors as

T. The results proved here are supported with suitable examples.

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