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The Estimation of Heavy Tails in Non-linear Models

Josephine N. Onyeka-Ubaka[1*] and Olaide Abass[2]
*[*1]Department of Statistics, University of Lagos, Akoka, Nigeria.*
[2]Department of Computer Science, Bells University, Ota, Nigeria.
Email addresses: jonyeka-ubaka@unilag.edu.ng; olaide.abass@yahoo.com
**Corresponding author*
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Abstract

A generalized student *t* distribution technique based on estimation of bilinear generalized autoregressive conditional heteroskedasticity (BL-GARCH) model is introduced. The paper investigates from empirical perspective, aspects of the model related to the economic and financial risk management and its impacts on volatility forecasting. The purposive sampling technique was applied to select four banks for the study, namely First Bank of Nigeria (FBN), Guaranty Trust Bank (GTB), United Bank for Africa (UBA) and Zenith Bank (ZEB). The four banks are selected, because their daily stock prices are considered to be more susceptible to volatility than those of other banks within the sampled period (January 2007–May 2022). The data collected were analyzed using MATLAB R2008b Software. The results show that the newly introduced generalized student *t* distribution is the most general of all the useful distributions applied in the BL-GARCH model parameter estimation. It serves as a general distribution for obtaining empirical characteristics such as volatility clustering, leptokurtosis and leverage effects between returns and conditional variances as well as capturing heavier and lighter tails in high frequency financial time series data.

Keywords: Leverage effect, BL-GARCH, Leptokurtosis, Heteroskedasticity, Nonlinear.

Introduction

Time varying parameter models have a long history in statistics. Modelling unequal variances in nonlinear time series is a challenging task. The ideas and techniques of modelling time series in these diverse areas of science and other related disciplines are proved to be useful and innovative (Box and Jenkins 1976, Park et al. 2023). Bilinear generalized autoregressive conditional heteroskedasticity (BL-GARCH) model is one of the major tools that statisticians and economists use to model financial markets' behaviour in the presence of political disorders, economic crises, wars or natural disasters. In such stress periods, prices of financial assets tend to fluctuate very profusely (Posedel 2005).

Statistically speaking, if the conditional variance of Y_t given $Y_{t-1}, Y_{t-2}, \cdots$ of a time series is not constant over time, then the process Y_t is conditionally heteroskedastic. Heteroskedasticity refers to the random errors having unequal variances. In particular, a heteroskedastic model has $Cov(e) = Diag(\sigma_t^2)$ (Christensen 1996). Let $Y = (Y_1, Y_2, \cdots, Y_n)$ be a Gaussian vector with mean vector μ and variance matrix $\sum$, where

$$\sum=\begin{bmatrix}\sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1n}\\ \sigma_{21} & \sigma_2^2 & \cdots & \sigma_{2n}\\ \vdots & \vdots & \vdots & \vdots\\ \sigma_{n1} & \sigma_{n2} & \cdots & \sigma_n^2\end{bmatrix} \quad (1)$$

If the expected value of all error terms when squared is the same at any given point, then the vector is homoskedastic (homogeneous). Therefore, $\sigma_1 = \sigma_2 = \cdots = \sigma_n$. When this assumption does not hold, the vector is heteroskedastic (Bollerslev et al. 1994, Box et al. 1994, Johnston and DiNardo 1997). There are several approaches to dealing with heteroskedasticity. If the error variance at different times is known, weighted regression is a good method. If as is the case with financial time series, the error variance is unknown, and must be estimated from the data, the changing error variance can be modelled with generalized autoregressive conditional heteroskedasticity model families using the bilinear generalized autoregressive conditional heteroskedasticity (BL-GARCH) model. One of the first authors to address variance changing over time was Mandelbrot (1963) who observed volatility clustering in modelling volatility that large (small) changes tend to be followed by large (small) changes of either sign. Klein (1977) used rolling estimates of quadratic residuals. Engle (1982) proposed autoregressive conditional heteroskedasticity (ARCH (q)) models that seemed to capture the empirical characteristics in the financial time series. These ARCH models have non-constant variances conditioned on the past which is a linear aggregate of recent past disturbances. This means that the more recent news will be the fundamental information that is relevant for modelling the present volatility with high frequency data.

Some leading scholars who studied ARCH model families include: Geweke (1986, 1989), Diebold and Nerlove (1989), Engle (1990), Jones et al. (1994), Gourieroux (2007) and Onyeka-Ubaka (2013). Harvey et al. (1992) established the existence of a few common factors explaining exchange rate volatility movements. Engle et al. (1990) showed that US bond treasury bills volatility changes are closely linked across maturities. These commonality of volatility changes hold not only across assets within a market, but also across different markets. For example, Barsky (1989) established that US stock and bond volatilities move together as stock prices fall, government bond prices go up, while Hamao et al. (1990) and Engle and Susmel (1993) discovered close links between volatility changes across international stock markets. Lamoureux and Lastrapes (1990) deduced that conditional heteroskedasticity may be caused by time dependence on the rate of information arrival to the market. They used the daily trading volume of stock markets as a proxy for such information arrival and confirm their significance. Mizrach (1990) associated ARCH models with the errors of the economic agent’s learning processes. Cai (1994) proposed the switching ARCH (SWARCH) model in which there were several different ARCH models which showed that the economy switched from one point to another following a Markov chain. In this model, there could be extremely high volatility process which was responsible for events such as the stock market crash in 2009. Gerald (2018) proposed estimating and forecasting West Africa stock market volatility using asymmetric GARCH models and concluded that the EGARCH model presented the best results in the analysis of the dynamics of market volatility behaviour. Ogundeji et al. (2021), on the other hand, used two Bayesian generalized autoregressive conditional heteroskedasticity (GARCH) models in capturing the volatility dynamics of the daily total cases and daily new cases of COVID-19 in Nigeria based on their conditional distributions. Their analyses revealed that the Bayesian Student *t* GARCH (1, 1) model performed better than the Bayesian Normal GARCH (1, 1) model. That volatilities move together should be encouraging to model builders, since they indicate that few common factors may explain much of the temporal variations in the conditional variances and covariances of asset returns. Storti and Vitale (2003) proposed BL-GARCH model in Gaussian framework. Diongue et al. (2010) extended their works using elliptical noise to capture the leverage effect or negative correlation between asset returns and volatilities. This paper, therefore, extends the works of Storti and Vitale (2003) to non-Gaussian framework using generalized student *t* distribution to capture lighter and heavier tails present in high frequency financial time series data.

Materials and Methods

The paper adapts the three recommended iterative steps of Box-Jenkins approach to select a suitable stochastic model (Box and Jenkins 1976, Box et al. 1994). The steps are: (i) Identification, (ii) Estimation, and (iii) Diagnostic checks.

The aim of the identification stage is to determine the transformation required to produce stationarity and also the order of autoregressive (AR) and moving average (MA) operators for the ε_t series. In a typical BL-GARCH modelling application, it is preferable that there are minimum of about 80 data points in the ε_t series in order to get reasonable maximum likelihood estimates for the parameters. This identification starts with time series plot which may reveal one of the following characteristics: (a) trends either in the mean level or variance of the time series (b) extreme values and outliers (c) seasonality. At the estimation stage, estimates are usually calculated for the conditional mean, autoregressive conditional hetroskedasticity, generalized autoregressive conditional hetroskedasticity and leverage effect parameters with the p-values and t-statistics as diagnostic checks. In this paper, natural logarithms are applied to obtain the transformation required to produce stationarity. The normality assumption of the residuals is usually not critical for obtaining good parameter estimates. As long as the ε_t's are independent and possess finite variance, reasonable estimates (Gaussian estimates) of the parameters can be obtained (Abass 1980, Onyeka-Ubaka and Anene 2020). Having observed that the conditional variance depends on the data, the paper uses maximum likelihood method (MLE) which is consistent and asymptotically normal. This is because financial time series data, for which BL-GARCH models are usually capable of capturing the characteristics, generate high frequency sampling of data. The paper uses the Gaussian (Normal) and the non-Gaussian (Generalized Student *t*) distributions to allow the model fit both the central part and the tails of the conditional distribution present in high frequency financial time series data. The elliptical normalized distributions (the Normal, Student *t* and the Generalized Student *t*) considered in this paper belong to exponential class. This is because their distributions can be expressed as

$$f(x,\theta) = p(\theta)\mathrm{B}(x)\exp[C(x)D(\theta)] \qquad (2)$$

for $-\infty < x < \infty$, for all θ in Θ and for a suitable choice of functions: $p(\cdot)$, $\mathrm{B}(\cdot)$, $C(\cdot)$ and $D(\cdot)$ defined to belong to the exponential family or exponential class, where $p(\theta)$ is purely a function of θ, $\mathrm{B}(x)$ is any function of *x*. Equation (2) is important in that any family of distribution that belongs to this class is complete and thus, there always exists a unique best estimator of θ. This implies that they have complete sufficient statistics for the estimators. The distributions are:

(a) The normal distribution is uniquely determined by its first two moments (Dallah et al. 2004). Hence, only the conditional mean and variance parameters enter the log-likelihood function

$$L(\theta) = -\frac{n}{2}\log(2\pi) - \frac{1}{2}\sum_{t=1}^{n}\left\{\log\left(\sigma_t^2 + \frac{\varepsilon_t^2}{\sigma_t^2}\right)\right\}, \qquad (3)$$

where n is the sample size. To obtain an analytical or numerical solution of the MLE, it is necessary to obtain the first derivative $\frac{\partial L(\theta)}{\partial \theta}$ and solve $\frac{\partial L(\theta)}{\partial \theta} = 0$. Assuming $\psi = \theta$, we have the score functions as

$$\frac{\partial L(\theta)}{\partial \theta} = \sum_{t=1}^{n} \frac{\varepsilon_t}{\sigma_t^2} \frac{\partial \mu}{\partial \theta} + \frac{1}{2} \sum_{t=1}^{n} \frac{1}{\sigma_t^2} \left(\frac{\varepsilon_t^2}{\sigma_t^2} - 1 \right) \frac{\partial \sigma_t^2}{\partial \theta} \quad (4)$$

Now, if the paper assumes that the innovations $(\varepsilon_t)_{t \in Z}$ have a conditional non-Gaussian:

(b) The Student t distribution is given as

$$f[z_t(\theta);\eta] = (2\pi)^{-1/2} (s^2)^{-1/2} \left(\frac{v}{2} \right)^{-1/2} \Gamma\left(\frac{v+1}{2} \right) \Gamma^{-1} \left(\frac{v}{2} \right) \left\{ 1 + \frac{(x-\mu)^2}{vs^2} \right\}^{-(v+1)/2} \quad (5)$$

(c) The Generalized Student t distribution is given as

$$f[z_t(\theta);\eta,q,\lambda_1,\lambda_2] = \frac{\Gamma\left(\frac{v+1+iq}{2} \right)\left(\frac{v+1-iq}{2} \right)}{(\lambda_1+\lambda_2) 2^{(1-v)} \pi (s)^{-\frac{1}{2}} \Gamma(v)} \left(1 + \frac{t^2}{v} \right)^{-(v+1)/2} \times \left(\lambda_1 \exp(q \ \arctan \frac{t}{v^{\frac{1}{2}}}) \right) + \left(\lambda_2 \exp(-q \ \arctan \frac{t}{v^{\frac{1}{2}}}) \right) \quad (6)$$

where $-\infty < t < \infty$, q is a complex number, $\lambda_1, \lambda_2 \geq 0$. λ_1 and λ_2 are the left and right tail parameters, respectively, v is the degree of freedom. The standardized deviate $t = (x-\mu)/s$ has distribution $t(0,\ 1,\ v)$, where x is the observations, μ is the mean and s is the standard deviation of the observations. The $\lambda_1, \lambda_2 \geq 0$ is a necessary condition because the probability density function must always be positive. Also, the normalization constant

$$\frac{\Gamma\left(\frac{v+1+iq}{2} \right)\left(\frac{v+1-iq}{2} \right)}{(\lambda_1+\lambda_2) 2^{(1-v)} \pi (s)^{-\frac{1}{2}} \Gamma(v)}$$

of Equation (6) is real, because the integrand is a real function on $(-\infty,\ \infty)$. It is clear that if $q = 0$ in Equation (6), the usual Student t distribution is derived. Moreover, for q = 0, the normalization constant of distribution Equation (6) is equal to the normalization constant of Student t distribution. The kurtosis of the Student t distribution is

$$\mathrm{E}\left[\varepsilon_t^4 \right] = \frac{3(v-2)}{v-4} \quad (7)$$

which is greater than three if v < 4. The MLE estimator $\hat{\theta}$ maximizes the log-likelihood function l_t given by

$$l_t = n\left[\log \Gamma \frac{(v+1)}{2} - \log \Gamma\left(\frac{v}{2} \right) - \frac{1}{2} \log \pi(v-2) \right] - \frac{1}{2} \sum_{t=1}^{n} \left\{ \log(\sigma_t^2) + (v+1) \log\left[1 + \frac{\varepsilon_t^2}{\sigma_t^2 (v-2)} \right] \right\} \quad (8)$$

where $2 < v \leq \infty$ and Γ is the Euler gamma function defined by $\Gamma(x) = \int_0^\infty t^{x-1} e^{-1} dt.$ When $v \to \infty$, we have the normal distribution, so that the smaller the value of v the fatter the tails.

This means that for large *v*, the product $\left(\frac{v}{2}\right)^{-1/2}\Gamma\left(\frac{v+1}{2}\right)\Gamma^{-1}\left(\frac{v}{2}\right)$ tends to unity, while the right-hand bracket in Equation (5) tends to $e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$. The score function is given by

$$\frac{\partial l_t}{\partial \theta} = \sum_{t=1}^{n}\left[\left(\frac{v+1}{v-2}\right)\frac{\varepsilon_t}{\sigma_t^2}\left(1+\frac{\varepsilon_t^2}{\sigma_t^2}\right)^{-1}\right]\frac{\partial \mu_t}{\partial \theta} + \frac{1}{2}\sum_{t=1}^{n}\left[\left(\frac{v+1}{v-2}\right)\frac{\varepsilon_t^2}{\sigma_t^2}\left(1+\frac{\varepsilon_t^2}{\sigma_t^2(v-2)}\right)^{-1} - 1\right]\frac{1}{\sigma_t^2}\frac{\partial \sigma_t^2}{\partial \theta} \quad (9)$$

and the Hessian matrix is given by

$$\begin{aligned}\frac{\partial^2 l_t}{\partial\theta\partial\theta'} &= \frac{v+1}{v-2}\sum_{t=1}^{n}\left(1+\frac{\varepsilon_t^2}{(v-2)\sigma_t^2}\right)^{-1}\frac{\varepsilon_t}{\sigma_t^2}\left[\frac{2}{v-2}\left(\frac{\varepsilon_t^2}{(v-2)\sigma_t^2}\right)^{-1}\frac{\varepsilon_t}{\sigma_t^2}-1\right]\frac{\partial \mu_t}{\partial \theta}\frac{\partial \mu_t}{\partial \theta'} \\
&+\frac{v+1}{v-2}\sum_{t=1}^{n}\left(1+\frac{\varepsilon_t^2}{(v-2)\sigma_t^2}\right)^{-1}\frac{\varepsilon_t}{\sigma_t^4}\left[\frac{1}{v-2}\left(1+\frac{\varepsilon_t^2}{(v-2)\sigma_t^2}\right)^{-1}\frac{\varepsilon_t^2}{\sigma_t^2}-1\right]\frac{\partial \mu_t}{\partial \theta}\frac{\partial \sigma_t^2}{\partial \theta'} \qquad (10)\\
&+\frac{v+1}{v-2}\sum_{t=1}^{n}\left(1+\frac{\varepsilon_t^2}{(v-2)\sigma_t^2}\right)^{-1}\frac{\varepsilon_t}{\sigma_t^4}\left[\frac{1}{v-2}\left(1+\frac{\varepsilon_t^2}{(v-2)\sigma_t^2}\right)^{-1}\frac{\varepsilon_t^2}{\sigma_t^2}-1\right]\frac{\partial \sigma_t^2}{\partial \theta}\frac{\partial \mu_t}{\partial \theta'} \\
&+\frac{1}{2}\sum_{t=1}^{n}\frac{1}{\sigma_t^4}\left[1+\frac{v+1}{v-2}\frac{\varepsilon_t^2}{\sigma_t^2}\left(1+\frac{\varepsilon_t^2}{(v-2)\sigma_t^2}\right)^{-1}\right]\left[\frac{\varepsilon_t^2}{(v-2)\sigma_t^2}\left(1-\frac{\varepsilon_t^2}{(v-2)\sigma_t^2}\right)^{-1}-2\right]\frac{\partial \sigma_t^2}{\partial \theta}\frac{\partial \sigma_t^2}{\partial \theta'}\end{aligned}$$

Results and Discussion

In this paper, a BL-GARCH model is adopted in which the innovation is composed of several sources of errors where each of the error sources has heteroskedastic specifications of the GARCH form. Since the error components cannot be separately observed given past observations, the independent variables in the variance equations are not measurable with respect to the available information set which complicates procedures. Following earlier work of Storti and Vitale (2003) and adopting Mohler (1973) non-linear representation of bilinear model, the state space representation of a bilinear model (of order m) in the control theory literatures is of the general form:

$$\begin{aligned}\underline{x}(t) &= \underline{A}\underline{x}(t-1) + \underline{B}\underline{\varepsilon}_t + \underline{C}\underline{x}(t-1)\underline{\varepsilon}_{t-1} \\ X_t &= \underline{H}\underline{x}(t)\end{aligned} \quad (11)$$

where the system matrix $\underline{A}$ and the input matrix $\underline{B}$ are square matrices of order $(m \times m)$; the state vector $\underline{x}$ and the control vector $\underline{\varepsilon}$ are column vectors of order $(m \times 1)$. The input $\underline{\varepsilon}$ is a usually unobservable random process and the systems coefficient matrices are to be estimated. If the paper nests the GARCH model and Equation (11), the BL-GARCH model is given as:

$$y_t = \mu + \varepsilon_t \quad (12)$$

$$\varepsilon_t^2 = \sigma_t^2 z_t^2 \quad (13)$$

$$\sigma_t^2 = \omega + \sum_{i=1}^{q}\alpha_i\varepsilon_{t-i}^2 + \sum_{j=1}^{p}\beta_j\sigma_{t-j}^2 + \sum_{k=1}^{r}c_k\sigma_{t-k}\varepsilon_{t-k} \quad (14)$$

If $\underline{c} = \underline{0}$, the model Equation (14) reduces to the state space representation of the GARCH model. In this sense, the bilinear generalized autoregressive conditional heteroskedasticity model is an asymmetric extension of the symmetric generalized autoregressive conditional heteroskedasticity model. Given the parameter space, Ω, and the expected parameter vector, $\omega = (\alpha, \theta)$, the assumption is that the parameter $\omega_0 \in \Omega \subseteq \mathbb{R}^{k+l+p+q+r+1}$ is in the interior of Ω, a compact parameter space. Specifically, for any vector $\omega \in \Omega$, we assume that

(a) The AR and MA polynomials have no common roots and that all their roots lie outside the unit circle.

$(b) \quad \alpha_0 > 0,\ \alpha_1, \alpha_2, \cdots, \alpha_q \geq 0 \ \ and \ \beta_1, \beta_2, \cdots \beta_p \geq 0.$

$(c) \quad c_1^2 < 4\alpha_i \beta_i, \quad for\ i = 1,\ 2, \cdots, r.$

$(d) \quad \sum_{i=1}^{q} \alpha_i + \sum_{j=1}^{p} \beta_j < 1.$

where the compact space is given as:

$$\Theta = \{\theta : \mu_1 \leq \mu \leq \mu_d,\ 0 < \omega_1 \leq \omega \leq \omega_d,\ 0 < \alpha_1 \leq \alpha \leq \alpha_d,\ 0 < \beta_1 \leq \beta \leq \beta_d\} \subseteq \{\theta : \mathrm{E}[\ln(\beta + \alpha z^2)] < 0\} \tag{15}$$

The generalized student *t* distribution with one skewness parameter and two tail parameters offers the study the potential to improve our ability to fit the data in the tail regions which are critical to the risk management and other financial economic applications. This is because downward movement of the markets is followed by higher volatilities than upward movement of the same magnitude (Pagan and Schwert 1990, Locke and Sayers 1993, Linton 1993, Muller and Yohai 2002, Eraker et al. 2003). So, it is important to use BL-GARCH (1, 1) model to capture asymmetric shocks to volatility. This distribution function will be acceptable if it converges to the probability density of the standard normal distribution and this leads to proposition 1.

Proposition 1

If $f\{z_t(\theta),\ \eta,\ q,\ \lambda_1,\ \lambda_2\}$ as in Equation (6) is distribution flexible, then it contains Pearson subordinate distributions.

Proof

The generalized student *t* distribution can be derived from a generalization of the Pearson differential equation as follows

$$\frac{df}{dz} = \frac{-g(z) f(z)}{h(z)} \tag{16}$$

where *g*(z) and *h*(z) are polynomials in the random variable *z*, and *f*(z) is the density function of *z*. In the standard Pearson system, *g*(z) is a polynomial in z of the degree less than or equal to one, whereas h(z) is a polynomial in z of degree less than or equal to two. The generalized solution of Equation (16) is

$$f(z) = \exp\left[-\int_0^z \left(\frac{g(s)}{h(s)} ds - \eta\right)\right], \quad z \in D \tag{17}$$

where the normalizing constant is given by

$$\eta = \ln \int \exp\left[-\int_0^z \left(\frac{g(s)}{h(s)}\right) ds\right] dz \tag{18}$$

The domain *D* of *f*(z) in Equation (17) is the open interval where *h*(z) is positive. The choice of $g(\cdot)$ and $h(\cdot)$ for the generalized student *t* distribution are:

$$g(z) = \sum_{i=0}^{\mathrm{M}-1} \alpha_i z^i \tag{19}$$

$$h(z) = \gamma^2 + z^2 \tag{20}$$

By substituting Equation (19) and Equation (20) in Equation (16), the generalized student *t* distribution is given by

$$f(z) = \exp\left[\lambda_1 \tan^{-1}\left(\frac{z}{\gamma}\right) + \lambda_2 \ln\left(\gamma^2 + z^2\right) + \sum_{i=3}^{\mathrm{M}} \lambda_i z^{i-2} - \eta\right] \quad -\infty < z < \infty \tag{21}$$

where $\eta = \log \int exp\left[\lambda_1 \tan^{-1}\left(\frac{z}{\gamma}\right) + \lambda_2 \ln\left(\gamma^2 + z^2\right) + \sum_{i=3}^{\mathrm{M}} \lambda_i z^{i-2}\right] dz$ (22)

And the distribution parameters λ_i are functions of the parameters $\{\alpha_i, \quad i = 0, 1, \cdots, \mathrm{M}-1;\ \gamma\}$ given in Equation (19) and Equation (20). Provided that in Equation (22), $\lambda_{\mathrm{M}} < 0$, all moments of the distribution exist. This distribution can exhibit a range of shapes including fat tails; sharp peaks and even multimodality (Lye et al. 1998). As the generalized student *t* distribution given by Equation (21) is derived from an extension of the Pearson exponential family, it directly contains many of the Pearson subordinate distributions as special cases. In particular, from the point of view of the existing ARCH models, these special cases include the Normal and Student *t* distributions. The standard normal distribution occurs when $\lambda_3 = -0.5$ and all remaining parameters are zero. The Student *t* distribution occurs when $\lambda_2 = -0.5(1+\gamma^2)$ and all remaining parameters are zero. A special case which turns out to be important in the empirical application is given by Equation (21) with $\lambda_i = 0,\ i > 2$ and $\lambda_2 = -0.5(1+\gamma^2)$.

$$f(z) = \exp\left[\lambda_1 \tan^{-1}\left(\frac{z}{\gamma}\right) - 0.5\left(1+\gamma^2\right)\ln\left(\gamma^2 + z^2\right) + \sum_{i=3}^{M} \lambda_i z^{i-2} - \tilde{\eta}\right] \quad -\infty < z < \infty \quad (23)$$

where $\tilde{\eta}$ is the normalized constant. This distribution is referred to as the skewed Student *t* distribution where skewness is controlled by the parameter, λ_1. When $\lambda_1 = 0$, there is no skewness and the distribution becomes Student *t* distribution.

This completes the proof.

Empirical study of real data

The adopted normalized elliptical distributions are empirically tested in four banks: First Bank of Nigeria (FBN), Guaranty Trust Bank (GTB), United Bank for Africa (UBA) and Zenith Bank (ZEB), selected by means of purposive sampling technique. These selected four banks are considered to be more susceptible to volatility than other banks and had passed the screening exercise conducted by the Central Bank of Nigeria (CBN) in August 2009. Their volumes of stocks traded on the floor of the Nigeria Stock Exchange (NSE) for the sampled period (January 2007–May 2022) were collected and analyzed using BL-GARCH model to observe the volatile nature of stocks within the sampled period. The series plots of the banks exhibit leptokurtosis and heavier tails as indicated in Figure 1.

The series returns show evidence of fat tails, the kurtosis is positive and evidence of skewness, which means that the tails are either heavier or lighter than the usual student *t* distribution. Looking at the plot of GTB stock prices, the student *t* distribution tends to infinity. This prompts the study to use the generalized student *t* distribution to capture the extreme values in Figure 2.

Figure 2 shows that the generalized student *t* distribution seems to be a more appropriate distribution for the selected banks data. The generalized student *t* distribution allows for situations where the tails are heavier or lighter than the usual student *t* distribution. The extreme values (left (λ_1) and right tails (λ_2)) for different banks are estimated automatically together with the plot of the P-P plots of the selected banks. The probability-probability plots of the four banks show that they are primarily a few large outliers that cause the departures of the system from normality. These departures are pointing out that there are other factors that interrupt the expected volatility of stock market prices of these banks on the floor of the Nigeria Stock Exchange. The factors may include among other things, giving loans to private and corporate firms to buy shares without due process, lack of strategic management and regular supervision.

The parameter results estimated by method of the maximum likelihood estimator using MATLAB (R2008b) software are given in Table 1.

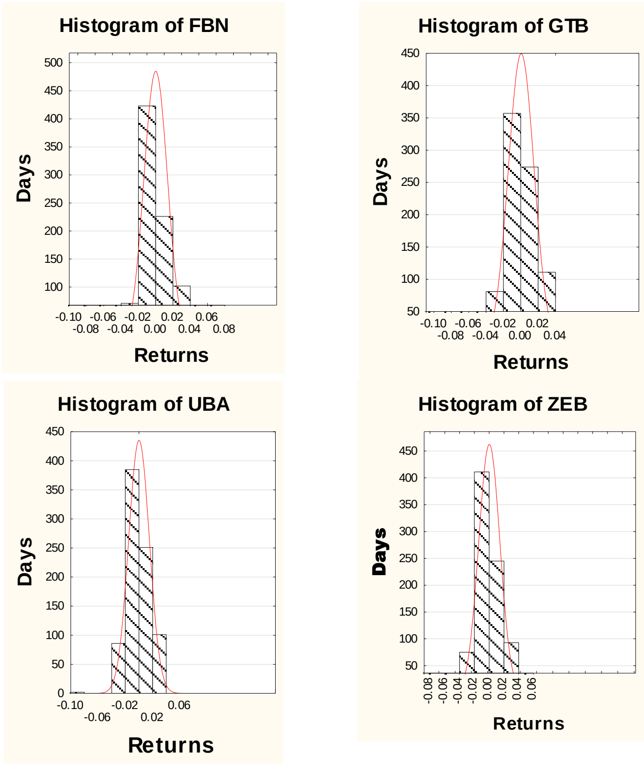

Figure 1: Histogram and student *t* distribution of the selected banks.

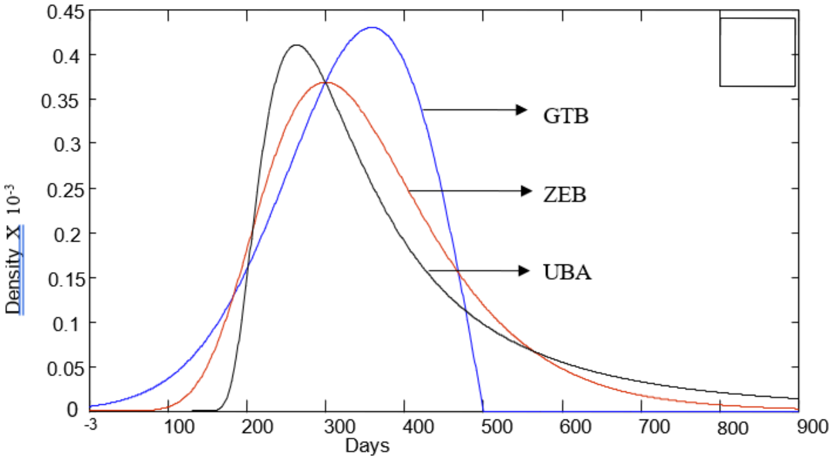

Figure 2: Plots of generalized student *t* distribution for banks that exhibit heavier and lighter tails.

Table 1: Conditional variance bilinear generalized autoregressive conditional heteroskedasticity (BL-GARCH (1, 1)) model parameter estimation results

	$\hat{\alpha}_0$	$\hat{\alpha}_1$	$\hat{\beta}_1$	$\hat{c}_1$	$\hat{\nu}$	$\hat{\lambda}_1$	$\hat{\lambda}_2$
Gaussian	-0.2950 0.01842* (0.01292, 22.4458)	0.2832 0.00352* (0.00823, 34.4107)	0.9511 0.01065* (0.01937 49.1017)	-0.0118 0.00537* (0.00568, -2.0775)	-		
Student *t*	-0.3002 0.02178* (0.05049, -5.9457)	0.4051 0.00413* (0.02279, 17.7753)	0.9238 0.00683* (0.07014, 13.1708)	-0.1036 0.01075* (0.01773, -5.8432)	3.3429 0.03937* (0.15028 22.2443)		
G. Std *t*	-0.3106 0.00572* (0.04512, -6.8839)	0.3916 0.00294* (0.02187, 17.9058)	0.9162 0.00158* (0.05218 17.5585)	-0.5314 0.02416* (0.08023, -6.6235)	3.5841 0.01927* (0.17209, 20.8269)	20.3	17.1
Gaussian FBN	1.8145e-^{005} 0.02481* (2.4118e^{06}, 7.5234)	0.17507 0.00322* (0.01720, 10.1778)	0.82493 0.00723* (0.01371, 60.1560)	0.00591 0.00537* (0.05316, 0.1112)	-		
Student *t* FBN	2e^{-007} 0.02178* (5.2309e^{-008}, 0.8234)	0.3634 0.00318* (0.04007, 9.0691)	0.6765 0.00659* (0.01941, 34.8531)	-0.0856 0.01075* (0.01575, -5.4349)	2.0682 0.05137* (0.1035, 19.9826)		
G. Std *t* FBN	2e-005 0.03217* (5.1308e^{-008}, 0.8234)	0.3831 0.00594* (0.04107, 9.3279)	0.6483 0.06253* (0.02394, 27.0802)	-0.0880 0.01075* (0.01455, -6.0481)	2.5618 0.05137* (0.1065, 24.05446)	23.4	10.8
GaussianGTB	-3.7376 0.02338* (0.69453, -5.3815)	0.7452 0.00813* (0.04746, 15.0695)	0.4548 0.05247* (0.09867, 4.6103)	-0.0760 0.05327* (0.03834, -1.9823)	-		
Student *t* GTB	-1.1341 0.02057* (0.29667, -3.8226)	0.4904 0.05164* (0.07519, 6.5221)	0.8391 0.06945* (0.04146, 2.0239)	-0.0283 0.06197* (0.05117, -5.5306)	2.2948 0.36092* (0.74276, 3.0896)		
G. Std *t* GTB	-3.1893 0.02178* (0.36262, -8.7952)	0.5190 0.01006* (0.05761, 9.0089)	0.4398 0.05352* (0.01567, 28.0664)	-0.0980 0.05169* (0.05212, -1.8803)	7.8974 0.06673* (0.84215, 9.3778)	18.4	7.8
Gaussian UBA	-1.8027 0.04358* (0.13484, -13.3690)	1.0215 0.01062* (0.05546, 18.4191)	0.7333 0.01965* (0.01753, 41.8311)	-0.1560 0.00951* (0.03737, -4.1745)	-		
Student *t* UBA	-0.4415 0.03761* (0.09842, -4.4859)	0.4915 0.00923* (0.04615, 10.6501)	0.94166 0.00714* (0.01326, 71.0078)	-0.1234 0.00714* (0.03633, -3.3966)	2.1625 0.03637* (0.31338, 6.9006)		
G. Std *t* UBA	-0.6134 0.02516* (0.18793, -3.2640)	0.4257 0.00623* (0.06183, 6.8850)	0.8419 0.00412* (0.03128, 26.9150)	-0.1760 0.01398* (0.02904, -6.0606)	5.9763 0.03637* (0.52160, 11.4576)	20.9	14.0
Gaussian ZEB	2.194e^{-005} 0.05329* (2.1776^{e-006}, 10.0754)	0.2285 0.01067* (0.01069, 21.3751)	0.7717 0.01594* (0.00894, 86.3199)	-0.0294 0.01553* (0.03008, -0.9774)	-		
Student *t* ZEB	2e^{-007} 0.04087* (5.9104^{e-008}, 3.3838)	0.3081 0.00771* (0.03615, 8.5228)	0.6918 0.01439* (0.02081, 33.2436)	-0.0189 0.03369* (0.02644, -0.7148)	3.6334 0.05213* (0.31416, 11.5654)		
G. Std *t* ZEB	2e^{-005} 0.02106* (5.8104^{e-008}, 4.3927)	0.3176 0.01194* (0.02581, 12.3058)	0.6543 0.00339* (0.02167, 30.1938)	-0.0297 0.03369* (0.01962, -1.5138)	5.6735 0.00168* (0.40267, 14.0897)	24.1	13.5

The asterisks (*) are the p-values. The values in parenthesis say (a, b) are the standard errors and *t*-statistics, respectively.

Table 1 represents conditional variance BL-GARCH (1, 1) model parameter estimation results. Results reveal that parameter estimates are satisfactory (asymptotically unbiased, efficient and consistent) in that the standard errors are small and the *t*-statistic for GARCH parameters (β) is high. It is clear from the analysis that estimate $\hat{\alpha}_1$ and $\hat{\beta}_1$ in the BL-GARCH (1, 1) model are significant at the 5% level with the volatility coefficient greater in magnitude. Hence, the hypothesis of constant variance is rejected, at least within sample period. Furthermore, the stationarity condition is satisfied for the three distributions, as $\hat{\alpha}_1 + \hat{\beta}_1 < 1$ at the maximum of the respective log-likelihood functions. Even when $\hat{\alpha}_1 + \hat{\beta}_1 > 1$, so long as $\mathrm{E}[\ln(\alpha_1 z_t^2 + \beta_1)] < 0$, covariance stationarity is established. The estimated asymmetric volatility response $\hat{c}_1$ is negative and significant for all models confirming the usual expectation in stock markets where downward movements (decreasing returns) are followed by higher volatility than upward movements (increasing returns). The results also follow the empirical findings of Storti and Vitale (2003), in that the kurtosis strongly depends on the leverage-effect response parameter. The results indicate that the BL-GARCH (1, 1) processes are appropriate for modelling the conditional variance of the selected banks returns. Using Akaike (1974), the BL-GARCH (1, 1) model with minimum AIC was selected as the best. The BL-GARCH (1, 1) conditional variance model that best fits the observed data is

$$\sigma_t^2 = -0.3106 + 0.3916\,\varepsilon_{t-1}^2 + 0.9162\,\sigma_{t-1}^2 - 0.5314\,\varepsilon_{t-1}\sigma_{t-1} \qquad (23)$$

where $\hat{\alpha}_0 = -0.3106$, $\hat{\alpha}_1 = ARCH(1) = 0.3916$, $\hat{\beta}_1 = GARCH(1) = 0.9162$ and $\hat{c}_1 = Leverage\ effect = -0.5314$

From the results obtained, the BL-GARCH (1, 1) model with Generalized Student *t* distribution fits GTB, UBA and ZEB data better while the First Bank of Nigeria data follows the Student *t* BL-GARCH (1, 1) models. This is because adding more parameters in modelling the FBN data does not improve the parameter estimates of the FBN. The parameter λ is therefore a good approximation of the degree up to which one is able to explain the variance/kurtosis of the disturbances. The GTB, UBA and ZEB series confirm these statements as seen in Figure 2.

Conclusion

The newly introduced generalized student *t* distribution in this paper is the most general of all the useful distributions applied in the BL-GARCH model parameter estimation. It offers a three parameter form which makes it more general than those available in the literature for obtaining empirical characteristics such as volatility clustering, leptokurtosis and leverage effect between returns and conditional variances as well as capturing heavier and lighter tails in high frequency financial time series data. The results of the extreme departure of some data are very crucial to risk managers for planning and decision-making processes. Thus, the empirical results of BL-GARCH model show that parameters can be evaluated from Gaussian and non-Gaussian distributions.

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Declaration of conflict of interest

There is no conflict of interest.

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