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Fixed Point Results for β-ψ- φ Contractive Mappings in $\mathcal{F}$-Metric Spaces with L-Fuzzy Mappings

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Abstract

In this paper, we explore the framework of $\mathcal{F}$-metric space, a well-known generalization of metric spaces, and establish fixed point results for β-ψ-φ contractive mappings in a complete $\mathcal{F}$-metric spaces endowed with L-fuzzy mappings. These additions broaden the body of knowledge in fixed point theory and fuzzy mappings. We showcase the practical applicability of our proposed results through illustrative examples and also, explore as an application, the solution for fuzzy initial-value problems.

Mathematics subject Classification 54H25, 47H10, 65Q30, 65Q10
Keywords Fixed point, $\mathcal{F}$-metric space, β-admissible, β-ψ- φ contraction mapping, L-fuzzy mappings

Introduction

The most basic outcome of metric fixed point theory is the classical Banach Contraction Principle, which is used to prove the existence of solutions to nonlinear integro-differential equations, Fredholm integral equations, and Volterra integral equations. However, many issues that arise in daily life are caused by incomplete data that is difficult to convey in traditional mathematics. In 1965, Zadeh (1965) introduced the notion of fuzzy sets, that proffer efficient means to handle imprecise information, laying the foundation for subsequent research in fuzzy mathematics. Building upon Zadeh's work, Goguen (1967) substituted the fully distributive lattice L for the interval [0, 1] by extending the idea of a fuzzy set to an L-fuzzy set. Heilpern (1981), further extended the notion of fuzzy mappings and derived fixed point results in the metric linear space. For more, we refer (Azam (2011), Abdullahi (2021), Ibrahim et al. (2024))

Rashid et al. (2014) introduced the conception of $\beta_{\mathcal{F}_L}$-admissible for two L-fuzzy mappings and derived numerous results for these mappings. Moreover, Jleli and Samet (2018) introduced a contemporary metric space, which is referred to as $\mathcal{F}$-metric space, to extend the classical metric space. In this direction, Alansari et al. (2020) used this concept and proved some fixed point results in $\mathcal{F}$-metric space with some open problems such as fixed point theorems for L-fuzzy mappings.

The existence of fixed point results was established by Samet et al. (2012) when they presented a novel class of contractive type mappings known as β-ψ contractive type mapping and β-admissible mappings in metric spaces. Further, Raji (2023) generalized the concept of β-ψ contractive type mappings and obtained various common fixed point results for this generalized class of contractive mappings. Further results can be found in

(Karapinar and Sameet 2012, Kumar et al. 2022, Pathak et al. 2023, Rashid et.al 2014, Raji et.al. 2024, Sanatee et al. 2023, Shahi et al. 2022).

Lateef (2024) recently proposed the idea of $\mathcal{F}$-metric space as a generalization of traditional metric space. They established several common fixed point theorems for (β-ψ)-contractions and proved the Banach contraction principle in the context of this generalized metric space. Based on the above insight, we establish fixed point results for β-ψ-φ contractive mappings in $\mathcal{F}$-metric spaces for L-fuzzy mappings within the framework of complete $\mathcal{F}$-metric spaces. To bolster our findings, we offer illustrative examples demonstrating the practical application of the presented results. Also, we explored as an application, the solution for fuzzy initial-value problems.

Materials and Methods

This study is purely theoretical, grounded in pure mathematics. The study will utilize mathematical concepts such as convergent sequences, operators, various types of mappings, abstract spaces, completeness, contraction mappings, iterative methods, Integrals, and differential equations. Definitions, lemmas, propositions, and theorems derived from these concepts will be employed to establish the main results.

Preliminaries

We begin this section by presenting the concept of β-ψ- φ contractive and β-admissible mappings.

Now, we denote Ψ the collections of nondecreasing functions $\Psi: [0,\infty) \to [0,\infty)$ such that

$$\sum_{n=1}^{+\infty} \psi^n(t) < \infty$$

for each $t > 0$, where ψ^n is the n-th iterate of ψ.

Definition 2.1 (Raji et al. 2024) Let X be a nonempty set, d be a metric such that (X,d) is a metric space and $T: X \to X$ be a mapping. T is referred to as β-ψ contractive if there exist a (c)-comparison functions $\psi \in \Psi$ and a function $\beta: X \times X \to \mathbb{R}$ such that

$$\beta(x,y)d(Tx,Ty) \leq \psi\big(d(x,y)\big), \text{ for all } x,y \in X. \quad (2.1)$$

Definition 2.2 (Karapinar and Samet 2012, Raji et al. 2004) Let X be a nonempty set, $T: X \to X$ and $\beta: X \times X \to \mathbb{R}^+$. Then, T is referred to as β-admisible mapping if for

$$x,y \in X, \beta(x,y) \geq 1 \Longrightarrow \beta(Tx,Ty) \geq 1. \quad (2.2)$$

Definition 2.3 Let Φ be the set of all function $\varphi: [0,\infty)^5 \to [0,\infty)$ satisfying the following:

(i) φ is continuous,

(ii) $\varphi(t_1,t_2,t_3,t_4,t_5) = 0$ if and only if $t_1t_2t_3t_4t_5 = 0$.

Example 2.4 The following function $\varphi: [0,\infty)^5 \to [0,\infty)$ belong to Φ.

(i) $\varphi(t_1,t_2,t_3,t_4,t_5) = t_1t_2t_3t_4t_5$.

(ii) $\varphi(t_1,t_2,t_3,t_4,t_5) = e^{t_1t_2t_3t_4t_5} - 1$.

(iii) $\varphi(t_1,t_2,t_3,t_4,t_5) = ln(1 + t_1t_2t_3t_4t_5)$.

Definition 2.5 A function T from a metric space (X,d) into itself is said to be β-ψ-φ contraction if there exist a function $\psi: [0,\infty) \to [0,\infty)$ and $\varphi: [0,\infty)^5 \to [0,\infty)$ satisfying

$$\beta(x,y)d(Tx,Ty) \leq \psi\big(d(x,y)\big) - \varphi\big(d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\big), \forall x,y \in X. \quad (2.3)$$

Definition 2.6 (Mohammed et al. 2021, Rashid et al. 2014) Let $L \neq \emptyset$ and $\preccurlyeq_L$ be partial order set and $(L, \preccurlyeq_L)$ be a partially ordered set.

(i) For any $x,y \in L, x \vee y \in L, x \wedge y \in L$, then L is referred to as lattice.

(ii) For any $\Omega \in L, \vee\Omega \in L, \ \wedge\Omega \in L$, then L referred to as complete lattice.

(iii) For any $x,y,z \in L$, $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z), x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$, then L is referred to as distributive lattice.

Definition 2.7 (Kanwal et al. 2022, Al Rawashdeh et al. 2018) Let L-fuzzy set Ω on a nonempty set X. A function $\Omega: X \to L$, where L satisfies (iii) of definition 2.6 with 1_L (top element) and 0_L(bottom element).
The α_L-level set of Ω is denoted by Ω_{α_L} define as

$$\Omega_{\alpha_L} = \{x: \alpha_L \preccurlyeq_L \Omega(x)\} \text{ if } \alpha_L \in L\backslash\{0_L\},$$
$$\Omega_{0_L} = \overline{\{x: 0_L \preccurlyeq_L \Omega(x)\}}.$$

Then, $\mathcal{F}_L(X)$ and cl(Ω) denotes L-fuzzy set on X and closure of Ω.
Define

$$X_{L_\wedge} = \begin{cases} 0_L, & if\ x \notin \Omega \\ 1_L, & if\ x \in \Omega, \end{cases}$$

The characteristic function $X_{L_\wedge}$ of L-fuzzy set Ω.
We now, introduce $\mathcal{F}$-metric space as follows:
Let $g: (0, +\infty) \to \mathbb{R}$. The set $\mathcal{F}$ is defined as the collection of functions g satisfying the following:
($\mathcal{F}_1$) $0 < x < t \Rightarrow g(x) \leq g(t)$,
($\mathcal{F}_2$) for the sequence $\{x_n\} \subseteq \mathbb{R}^+$, $\lim_{n\to\infty} x_n = 0 \Leftrightarrow \lim_{n\to\infty} g(x_n) = -\infty$.
Definition 2.8 (Jleli and Samet 2018) Let X be nonempty set and $d_\mathcal{F}: X \times X \to [0, +\infty)$. Suppose there exists $(g, h) \in \mathcal{F} \times [0, +\infty)$ such that
(i) $(x, y) \in X \times X, d_\mathcal{F}(x, y) = 0 \Leftrightarrow x = y$,
(ii) $d_\mathcal{F}(x, y) = d_\mathcal{F}(y, x)$, for all $(x, y) \in X \times X$,
(iii) for every $(x, y) \in X \times X$, for every $N \in \mathbb{N}, N \geq 2$, and for each $(u_i)_{i=1}^N \subset X$ with $(u_1, u_N) = (x, y)$, we have

$$d_\mathcal{F}(x, y) > 0 \Rightarrow g\big(d_\mathcal{F}(x, y)\big) \leq g[\textstyle\sum_{i=1}^{N-1} d_\mathcal{F}(x_i, x_{i+1})] + h.$$

Then, $d_\mathcal{F}$ is referred to as $\mathcal{F}$-metric on X and $(X, d_\mathcal{F})$ called $\mathcal{F}$-metric space.
Example 2.9 (Jleli and Samet 2018) Let $d_\mathcal{F}: \mathbb{R} \times \mathbb{R} \to [0, +\infty)$ be a function defined by

$$d_\mathcal{F}(x, y) = \begin{cases} (x-y)^2, & if\ (x, y) \in [0,3] \times [0,3] \\ |x - y|, & if\ (x, y) \notin [0,3] \times [0,3], \end{cases}$$

with $g(t) = \ln(t)$ and $h = \ln(3)$, is $\mathcal{F}$-metric.
Definition 2.10 (Jleli and Samet 2018) Let $(X, d_\mathcal{F})$ be $\mathcal{F}$-metric space.
(i) Let $\{x_n\} \subseteq X$. The sequence $\{x_n\}$ is referred to as $\mathcal{F}$-convergent to $x \in X$ if $\{x_n\}$ is convergent to x in $\mathcal{F}$-metric $d_\mathcal{F}$.
(ii) The sequence $\{x_n\}$ is referred to as $\mathcal{F}$-Cauchy, iff

$$\lim_{n,m\to\infty} d_\mathcal{F}(x_n, x_m) = 0.$$

(iii) If for each $\mathcal{F}$-Cauchy sequence in X is $\mathcal{F}$-convergent to $x \in X$, then $(X, d_\mathcal{F})$ is $\mathcal{F}$-complete.
Lemma 2.11 (Jleli and Samet 2018) Assume X_1 and X_2 be nonempty compact subsets of $\mathcal{F}$-metric space $(X, d_\mathcal{F})$ that is closed, if $x \in X_1$, then

$$d_\mathcal{F}(x, X_2) \leq H_\mathcal{F}(X_1, X_2),$$ where $H_F(X_1, X_2)$ denotes the Hausdorff distance between the sets X_1 and X_2, defined as the greatest distance between any point in one set to the closest point in the other set.
Lemma 2.12 (Lateef 2024) Let $(X, d_\mathcal{F})$ be $\mathcal{F}$-metric space and X_1 be nonempty closed subsets of X and $q > 1$. Then, for each $x \in X$ with $d_\mathcal{F}(x, X_1) > 0$, there exists $y \in X_1$ such that

$$d_\mathcal{F}(x, y) < q d_\mathcal{F}(x, X_1).$$

Definition 2.13 (Raji and Ibrahim 2024) Let $(X, d_\mathcal{F})$ be $\mathcal{F}$-metric space and $\beta: (X, d_\mathcal{F}) \times (X, d_\mathcal{F}) \to [0, +\infty)$. Let T, f be a pair of fuzzy mappings from X into $\mathcal{F}_L(X)$. Then, the pair (T, f) is referred to as an $\alpha_\mathcal{F}$-admissible if:
(i) for a point $x \in X$ and $y \in [Tx]_{\alpha_T(x)}$, where $\alpha_T(x) \in (0,1]$ with $\beta(x, y) \geq 1$, then, $\beta(y, z) \geq 1$, for all $z \in [fy]_{\alpha_f(y)} \neq \emptyset$ where $\alpha_f(y) \in (0,1]$,

(ii) for a point $x \in X$ and $y \in [fx]_{\alpha_T(x)}$, where $\alpha_f(x) \in (0,1]$ with $\beta(x,y) \geq 1$, then, $\beta(y,z) \geq 1$, for all $z \in [Ty]_{\alpha_T(y)} \neq \emptyset$ where $\alpha_T(y) \in (0,1]$.

Main Results

Theorem 3.1. Let $(X, d_{\mathcal{F}})$ be $\mathcal{F}$-metric space and $\beta: (X, d_{\mathcal{F}}) \times (X, d_{\mathcal{F}}) \to [0, \infty)$. Let T be a L-fuzzy mapping from $(X, d_{\mathcal{F}})$ into $\mathcal{F}_L(X)$ satisfying the following:
(i) $(X, d_{\mathcal{F}})$ is $\mathcal{F}$-complete,
(ii) for a point $x_0 \in X$ there exists $\alpha_L(x) \in L\backslash\{0_L\}$ such that $x_1 \in [Tx_0]_{\alpha_{\mathcal{F}}(x_0)}$ with $\beta(x_0, x_1) \geq 1$,
(iii) for all $x, y \in$ X, there exist $\psi \in \Psi$ and $\varphi \in \Phi$ such that

$$\beta(x,y)H_{\mathcal{F}}\left([Tx]_{\alpha_L(x)}, [Ty]_{\alpha_L(y)}\right) \leq \psi\left(d_{\mathcal{F}}(x,y)\right) - \varphi\left(d(x,y), d\left(x, [Tx]_{\alpha_L(x)}\right), d\left(y, [Ty]_{\alpha_L(y)}\right), d\left(x, [Ty]_{\alpha_L(y)}\right), d\left(y, [Tx]_{\alpha_L(x)}\right)\right), \quad (3.1)$$

(iv) T is $\beta_{\mathcal{F}}$-admissible,
(v) for all n, $\{x_n\}$ is a sequence in X such that $\beta(x_n, x_{n+1}) \geq 1$ and $x_n \to x \in X$ as $n \to \infty$, then there exists $\beta(x_n, x) \geq 1$ for all n.
Then, T has a fixed point $x^* \in [Tx^*]_{\alpha_{\mathcal{F}}(x^*)}$.

Proof By condition (ii), we can choose a point $x_0 \in X$, there exists $\alpha_L(x_0) \in L\backslash\{0_L\}$ such that $[Tx_0]_{\alpha_L(x_0)}$ is nonempty and there exists a point $x_1 \in [Tx_0]_{\alpha_L(x_0)}$ with $\beta(x_0, x_1) \geq 1$. Again, for x_1, there exists $\alpha_L(x_1) \in L\backslash\{0_L\}$ such that $[Tx_1]_{\alpha_L(x_1)} \in CB(X)$.
With (3.1) and Lemma 2.11, we have

$$\begin{aligned} 0 &< d_{\mathcal{F}}\left(x_1, [Tx_1]_{\alpha_L(x_1)}\right) \\ &\leq H_{\mathcal{F}}\left([Tx_0]_{\alpha_L(x_0)}, [Tx_1]_{\alpha_L(x_1)}\right) \\ &\leq \beta(x_0, x_1) H_{\mathcal{F}}\left([Tx_0]_{\alpha_L(x_0)}, [Tx_1]_{\alpha_L(x_1)}\right) \\ &\leq \psi\left(d_{\mathcal{F}}(x_0, x_1)\right) - \varphi\left(d(x_0,x_1), d\left(x_0, [Tx_0]_{\alpha_L(x_0)}\right), d\left(x_1, [Tx_1]_{\alpha_L(x_1)}\right), d\left(x_0, [Tx_1]_{\alpha_L(x_1)}\right), d\left(x_1, [Tx_0]_{\alpha_L(x_0)}\right)\right) \\ &\leq \psi\left(d_{\mathcal{F}}(x_0, x_1)\right) - \varphi\left(d(x_0,x_1), d(x_0,x_1), d\left(x_1, [Tx_1]_{\alpha_L(x_1)}\right), d\left(x_0, [Tx_1]_{\alpha_L(x_1)}\right), d(x_1,x_1)\right) \end{aligned}$$

By Definition 2.3, we obtain

$$\leq \psi\left(d_{\mathcal{F}}(x_0, x_1)\right) \quad (3.2)$$

Then, by Lemma 2.12, for $q > 1$, there exists $x_2 \in [Tx_1]_{\alpha_L(x_1)}$ such that

$$0 < d_{\mathcal{F}}(x_1, x_2) < q d_{\mathcal{F}}\left(x_1, [Tx_1]_{\alpha_L(x_1)}\right). \quad (3.3)$$

Considering (3.2) and (3.3), we get

$$0 < d_{\mathcal{F}}(x_1, x_2) \leq q\psi\left(d_{\mathcal{F}}(x_0, x_1)\right). \quad (3.4)$$

Obviously, $x_1 \neq x_2$, as $d_{\mathcal{F}}(x_1, x_2) < q\psi\left(d_{\mathcal{F}}(x_0, x_1)\right)$. Since ψ is strictly increasing, so $\psi\left(d_{\mathcal{F}}(x_1, x_2)\right) < \psi\left(q\psi\left(d_{\mathcal{F}}(x_0, x_1)\right)\right)$.

Set $q_1 = \frac{\psi\left(q\psi\left(d_{\mathcal{F}}(x_0,x_1)\right)\right)}{\psi\left(d_{\mathcal{F}}(x_1,x_2)\right)}$. Then, $q_1 > 1$. Again, for $x_2 \in X$, there exists $\alpha_L(x_2) \in L\backslash\{0_L\}$ such that $[Tx_2]_{\alpha_L(x_2)}$ is nonempty. Now, let $x_2 \notin [Tx_2]_{\alpha_L(x_2)}$. Since $\beta(x_0, x_1) \geq 1$ and by (iv), $\beta(x_1, x_2) \geq 1$. With (3.1) and Lemma 2.11, we have

$$\begin{aligned} 0 &< d_{\mathcal{F}}\left(x_2, [Tx_2]_{\alpha_L(x_2)}\right) \\ &\leq H_{\mathcal{F}}\left([Tx_1]_{\alpha_L(x_1)}, [Tx_2]_{\alpha_L(x_2)}\right) \\ &\leq \beta(x_1, x_2) H_{\mathcal{F}}\left([Tx_1]_{\alpha_L(x_1)}, [Tx_2]_{\alpha_L(x_2)}\right) \\ &\leq \psi\left(d_{\mathcal{F}}(x_1, x_2)\right) - \varphi\left(d(x_1,x_2), d\left(x_1, [Tx_1]_{\alpha_L(x_1)}\right), d\left(x_2, [Tx_2]_{\alpha_L(x_2)}\right), d\left(x_1, [Tx_2]_{\alpha_L(x_2)}\right), d\left(x_2, [Tx_1]_{\alpha_L(x_1)}\right)\right) \\ &\leq \psi\left(d_{\mathcal{F}}(x_1, x_2)\right) - \end{aligned}$$

$\varphi\left(d(x_1,x_2), d\left(x_1,[Tx_1]_{\alpha_L(x_1)}\right), d(x_2,x_1), d(x_1,x_1), d\left(x_2,[Tx_1]_{\alpha_L(x_1)}\right)\right)$ (3.5)

From definition 2.3, we have (3.5) as

$$\leq \psi\big(d_{\mathcal{F}}(x_1,x_2)\big)$$

By Lemma 2.12, for $q_1 > 1$, there exists $x_3 \in [Tx_2]_{\alpha_L(x_2)}$ such that

$$0 < d_{\mathcal{F}}(x_2,x_3) < q_1 d_{\mathcal{F}}\big(x_2,[Tx_2]_{\alpha_L(x_2)}\big). \quad (3.6)$$

Consider (3.5) and (3.6), we get

$$0 < d_{\mathcal{F}}(x_2,x_3) \leq q_1\psi\big(d_{\mathcal{F}}(x_1,x_2)\big) = \psi\left(q\psi\big(d_{\mathcal{F}}(x_0,x_1)\big)\right). \quad (3.7)$$

Obviously, $x_2 \neq x_3$, as $d_{\mathcal{F}}(x_2,x_3) < \psi\left(q\psi\big(d_{\mathcal{F}}(x_0,x_1)\big)\right)$. Since ψ is strictly increasing, so $\psi\big(d_{\mathcal{F}}(x_2,x_3)\big) < \psi^2\left(q\psi\big(d_{\mathcal{F}}(x_0,x_1)\big)\right)$.

Set $q_2 = \frac{\psi^2\left(q\psi(d_{\mathcal{F}}(x_0,x_1))\right)}{\psi(d_{\mathcal{F}}(x_2,x_3))}$. Then, $q_2 > 1$. Again, for $x_3 \in X$, there exists $\alpha_L(x_3) \in L\backslash\{0_L\}$ such that $[Tx_3]_{\alpha_L(x_3)}$ is nonempty. Now, let $x_3 \notin [Tx_3]_{\alpha_L(x_3)}$. Since $\beta(x_1,x_2) \geq 1$ and by (iv), $\beta(x_2,x_3) \geq 1$. With (3.1) and Lemma 2.11, we have

$$\begin{aligned} 0 &< d_{\mathcal{F}}\big(x_3,[Tx_3]_{\alpha_L(x_3)}\big) \\ &\leq H_{\mathcal{F}}\big([Tx_2]_{\alpha_L(x_2)},[Tx_3]_{\alpha_L(x_3)}\big) \\ &\leq \beta(x_2,x_3)H_{\mathcal{F}}\big([Tx_2]_{\alpha_L(x_2)},[Tx_3]_{\alpha_L(x_3)}\big) \\ &\leq \psi\big(d_{\mathcal{F}}(x_2,x_3)\big) - \end{aligned}$$

$$\varphi\left(d(x_2,x_3), d\big(x_2,[Tx_2]_{\alpha_L(x_2)}\big), d\big(x_3,[Tx_3]_{\alpha_L(x_3)}\big), d\big(x_2,[Tx_3]_{\alpha_L(x_3)}\big), d\big(x_3,[Tx_2]_{\alpha_L(x_2)}\big)\right)$$

$$\leq \psi\big(d_{\mathcal{F}}(x_2,x_3)\big) - \varphi\left(d(x_2,x_3), d(x_2,x_3), d\big(x_3,[Tx_3]_{\alpha_L(x_3)}\big), d\big(x_2,[Tx_3]_{\alpha_L(x_3)}\big), d(x_3,x_3)\right)$$

Again, by Definition 2.3, we have

$$\leq \psi\big(d_{\mathcal{F}}(x_2,x_3)\big) \quad (3.8)$$

By Lemma 2.12, for $q_2 > 1$, there exists $x_4 \in [Tx_3]_{\alpha_L(x_3)}$ such that

$$0 < d_{\mathcal{F}}(x_3,x_4) < q_2 d_{\mathcal{F}}\big(x_3,[Tx_3]_{\alpha_L(x_3)}\big). \quad (3.9)$$

Consider (3.8) and (3.9), we get

$$0 < d_{\mathcal{F}}(x_3,x_4) \leq \psi^2\left(q\psi\big(d_{\mathcal{F}}(x_0,x_1)\big)\right). \quad (3.10)$$

Continuing this process having chosen $x_1, x_2, x_3, x_4 \ldots$, we establish a sequence $\{x_n\}$ in X such that $x_{2n+1} \in [Tx_{2n}]_{\alpha_L(x_{2n})}, x_{2n+2} \in [Tx_{2n+1}]_{\alpha_L(x_{2n+1})}$ and $\beta(x_{n-1},x_n) \geq 1$, then for all n, we have

$$d_{\mathcal{F}}(x_{2n+1},x_{2n+2}) \leq \psi^{2n}\left(q\psi\big(d_{\mathcal{F}}(x_0,x_1)\big)\right) \quad (3.11)$$

and

$$d_{\mathcal{F}}(x_{2n+2},x_{2n+3}) \leq \psi^{2n+1}\left(q\psi\big(d_{\mathcal{F}}(x_0,x_1)\big)\right) \quad (3.12)$$

From (3.11) and (3.12), we get

$$d_{\mathcal{F}}(x_n,x_{n+1}) \leq \psi^{n-1}\left(q\psi\big(d_{\mathcal{F}}(x_0,x_1)\big)\right), \quad (3.13)$$

implies for $m > n$,

$$\sum_{i=n}^{m-1} d_{\mathcal{F}}(x_n,x_{n+1}) \leq \sum_{i=n}^{m-1} \psi^{n-1}\left(q\psi\big(d_{\mathcal{F}}(x_0,x_1)\big)\right).$$

(3.14)

Let $\varepsilon > 0$ and $n(\varepsilon) \in \mathbb{N}$ such that $\sum_{n \geq n(\delta)} \psi^{n-1}\left(q\psi\big(d_{\mathcal{F}}(x_0,x_1)\big)\right) < \varepsilon$. Let $(g,h) \in \mathcal{F} \times [0,+\infty)$ be such that (iii) of definition 2.8 is satisfied. Again, let $\varepsilon > 0$ be fixed. By $\mathcal{F}_1$, there exists $\delta > 0$ such that

$$0 < t < \delta \Rightarrow g(t) < g(\delta) - h. \quad (3.15)$$

Now, consider (3.13), (3.14) and $\boldsymbol{\mathcal{F}}_1$, we have for all $m > n \geq N$,
$g(\sum_{i=n}^{m-1} d_{\mathcal{F}}(x_i, x_{i+1})) \leq$
$$g\left(\sum_{i=n}^{m-1} \psi^{i-1}\left(q\psi\left(d_{\mathcal{F}}(x_0, x_1)\right)\right)\right) \leq g\left(\sum_{n\geq n(\delta)} \psi^{i-1}\left(q\psi\left(d_{\mathcal{F}}(x_0, x_1)\right)\right)\right) < g(\varepsilon) - h. \quad (3.16)$$
By (iii) of Definition 2.8 and (3.16), we get
$$d_{\mathcal{F}}(x_n, x_m) > 0, \quad (3.17)$$
implies
$$g\left(d_{\mathcal{F}}(x_n, x_m)\right) \leq g\left(\sum_{i=n}^{m-1} d_{\mathcal{F}}(x_i, x_{i+1})\right) + h < g(\varepsilon). \quad (3.18)$$
By $\boldsymbol{\mathcal{F}}_2$, we have $d_{\mathcal{F}}(x_n, x_m) < \varepsilon, m > n \geq N$. It follows that the sequence $\{x_n\}$ is $\mathcal{F}$-Cauchy. Since $(X, d_{\mathcal{F}})$ is $\boldsymbol{\mathcal{F}}$-complete, there exists $x^* \in X$ such that the sequence $\{x_n\}$ is $\mathcal{F}$-convergence to x^*, that is,
$$\lim_{n\to\infty} d_{\mathcal{F}}(x_n, x^*) = 0. \quad (3.19)$$
To show that $x^* \in [Tx^*]_{\alpha_L(x^*)}$, we let $d_{\mathcal{F}}\left(x^*, [Tx^*]_{\alpha_L(x^*)}\right) > 0$. Since T is $\beta_{\mathcal{F}}$-admissible, $\beta(x_{2n-1}, x^*) \geq 1$ for all $n \in \mathbb{N}$.
By Definition of g and (iii) of Definition 2.8, we have
$$g\left(d_{\mathcal{F}}\left(x^*, [Tx^*]_{\alpha_L(x^*)}\right)\right) \leq g\left(d_{\mathcal{F}}(x^*, x_{2n}) + d_{\mathcal{F}}\left(x_{2n}, [Tx^*]_{\alpha_L(x^*)}\right)\right) + h$$
$$\leq g\left(d_{\mathcal{F}}(x^*, x_{2n}) + H_{\mathcal{F}}\left([Tx_{2n-1}]_{\alpha_L(x_{2n-1})}, [Tx^*]_{\alpha_L(x^*)}\right)\right) + h$$
$$\leq g\left(d_{\mathcal{F}}(x^*, x_{2n}) + \beta(x_{2n-1}, x^*)H_{\mathcal{F}}\left([Tx_{2n-1}]_{\alpha_L(x_{2n-1})}, [Tx^*]_{\alpha_L(x^*)}\right)\right) + h$$
$$\leq g\Big(d_{\mathcal{F}}(x^*, x_{2n}) + \psi\left(d_{\mathcal{F}}(x^*, x_{2n-1})\right) -$$
$$\varphi\left(d(x^*, x_{2n}), d\left(x^*, [Tx^*]_{\alpha_L(x^*)}\right), d\left(x_{2n}, [Tx_{2n}]_{\alpha_L(x_{2n})}\right), d\left(x^*, [Tx_{2n}]_{\alpha_L(x_{2n})}\right), d\left(x_{2n}, [Tx^*]_{\alpha_L(x^*)}\right)\right)\Big) + h$$
By Definition 2.3, we have
$$\leq g\left(d_{\mathcal{F}}(x^*, x_{2n}) + d_{\mathcal{F}}(x^*, x_{2n-1})\right) + h. \quad (3.20)$$
On taking limit in (3.20), using (3.19) and $\boldsymbol{\mathcal{F}}_2$, we have
$$\lim_{n\to\infty} g\left(d_{\mathcal{F}}\left(x^*, [Tx^*]_{\alpha_L(x^*)}\right)\right) = \lim_{n\to\infty} g\left(d_{\mathcal{F}}(x^*, x_{2n}) + d_{\mathcal{F}}(x^*, x_{2n-1})\right) + h = -\infty,$$
a contradiction. Hence, $d_{\mathcal{F}}\left(x^*, [Tx^*]_{\alpha_L(x^*)}\right) = 0$, which implies $x^* \in [Tx^*]_{\alpha_L(x^*)}$.
Thus, T has a fixed point.

Example 3.2. Consider $X = [0,1]$, for all $x, y \in X$, $\boldsymbol{\mathcal{F}}$-metric $d_{\mathcal{F}}: X \times X \to \mathbb{R}_0^+$ is define $d_{\mathcal{F}}(x, y) = |x - y|$, and for $t > 0$ and $h = 0$, $g(t) = \ln(t)$. Then, $(X, d_{\mathcal{F}})$ is $\boldsymbol{\mathcal{F}}$-complete $\boldsymbol{\mathcal{F}}$-metric space. Assume $L = \{w_1, w_2, w_3, w_4\}$ with $w_1 \preccurlyeq_L w_2 \preccurlyeq_L \preccurlyeq w_4$ and $w_1 \preccurlyeq_L w_3 \preccurlyeq_L \preccurlyeq w_4$, where w_2 and w_3 are not comparable. We define $T: X \to \boldsymbol{\mathcal{F}_L}(X)$ as
$$T(x)(t) = \begin{cases} w_4, & if\ 0 < t \leq \frac{x}{6} \\ w_2, & if\ \frac{x}{6} < t \leq \frac{x}{3} \\ w_{3,} & if\ \frac{x}{3} < t \leq \frac{x}{2} \\ w_1, & if\ \frac{x}{2} < t \leq 1, \end{cases}$$
For all $x \in$ X, there exists $\alpha_L(x) = w_4$, such that
$$[Tx]_{\alpha_L(x)} = \left[0, \frac{x}{6}\right]$$

Hence, all the conditions of Theorem 3.1 are satisfied with $\psi(t) = \frac{1}{2}t$, $\varphi(t) = \frac{1}{3}t$, for $t > 0$.
Thus, there exists $0 \in [0,1]$, that is, $0 \in [T0]_{\alpha_L(0)}$.

Corollary 3.3 Let $(X, d_{\mathcal{F}})$ be $\mathcal{F}$-metric space and $\beta: (X, d_{\mathcal{F}}) \times (X, d_{\mathcal{F}}) \to [0, \infty)$. Let T, f be a pair of L-fuzzy mappings from $(X, d_{\mathcal{F}})$ into $\mathcal{F}_L(X)$ satisfying the following:
(i) $(X, d_{\mathcal{F}})$ is $\mathcal{F}$-complete,
(ii) for a point $x_0 \in X$, there exists $\alpha_L(x) \in L\backslash\{0_L\}$ such that $x_1 \in [Tx_0]_{\alpha_L(x_0)}$ or $x_1 \in [fx_0]_{\alpha_L(x_0)}$ with $\beta(x_0, x_1) \geq 1$,
(iii) for all $x, y \in$ X, there exist $\psi \in \Psi$ and $\varphi \in \Phi$ such that

$$\beta(x,y)H_{\mathcal{F}}\left([Tx]_{\alpha_L(x)}, [fy]_{\alpha_L(y)}\right) \leq \psi\left(d_{\mathcal{F}}(x,y)\right) - \varphi\left(d(x,y), d\left(x, [Tx]_{\alpha_L(x)}\right), d\left(y, [fy]_{\alpha_L(y)}\right), d\left(x, [fy]_{\alpha_L(y)}\right), d\left(y, [Tx]_{\alpha_L(x)}\right)\right), \quad (3.21)$$

(iv) (T, f) is $\beta_{\mathcal{F}}$-admissible,
(v) for all n, $\{x_n\}$ is a sequence in X such that $\beta(x_n, x_{n+1}) \geq 1$ and $x_n \to x \in X$ as $n \to \infty$, then there exists $\beta(x_n, x) \geq 1$ for all n.
Then, T and f have a common fixed point $x^* \in [Tx^*]_{\alpha_L(x^*)} \cap [fx^*]_{\alpha_L(x^*)}$.

Proof The result follow from Theorem 3.1

Corollary 3.4 Let $(X, d_{\mathcal{F}})$ be $\mathcal{F}$-metric space and $\beta: (X, d_{\mathcal{F}}) \times (X, d_{\mathcal{F}}) \to [0, \infty)$. Let T be fuzzy mapping from $(X, d_{\mathcal{F}})$ into $\mathcal{F}(X)$ satisfying the following:
(i) $(X, d_{\mathcal{F}})$ is $\mathcal{F}$-complete,
(ii) for a point $x_0 \in X$, there exists $\alpha(x) \in (0,1]$ such that $x_1 \in [Tx_0]_{\alpha(x_0)}$ with $\beta(x_0, x_1) \geq 1$,
(iii) for all $x, y \in$ X, there exist $\psi \in \Psi$ and $\varphi \in \Phi$ such that

$$\beta(x,y)H_{\mathcal{F}}\left([Tx]_{\alpha(x)}, [Ty]_{\alpha(y)}\right) \leq \psi\left(d_{\mathcal{F}}(x,y)\right) - \varphi\left(d(x,y), d\left(x, [Tx]_{\alpha_L(x)}\right), d\left(y, [Ty]_{\alpha_L(y)}\right), d\left(x, [Ty]_{\alpha_L(y)}\right), d\left(y, [Tx]_{\alpha_L(x)}\right)\right), \quad (3.22)$$

(iv) T is $\beta_{\mathcal{F}}$-admissible,
(v) for all n, $\{x_n\}$ is a sequence in X such that $\beta(x_n, x_{n+1}) \geq 1$ and $x_n \to x \in X$ as $n \to \infty$, then there exists $\beta(x_n, x) \geq 1$ for all n.
Then, T has a fixed point $x^* \in [Tx^*]_{\alpha(x^*)}$.

Proof Let L-fuzzy mappings $A: X \to \mathcal{F}_L(X)$ be defined as

$$Ax = X_{L_{Tx}}.$$

Then, for all $\alpha_L(x) \in L\backslash\{0_L\}$, we get

$$[Ax]_{\alpha_L(x)} = Tx.$$

Hence, the remaining proof follows from Theorem 3.1.

Corollary 3.5 Let $(X, d_{\mathcal{F}})$ be $\mathcal{F}$-metric space and $\beta: (X, d_{\mathcal{F}}) \times (X, d_{\mathcal{F}}) \to [0, \infty)$. Let T, f be a pair of fuzzy mappings from $(X, d_{\mathcal{F}})$ into $\mathcal{F}(X)$ satisfying the following:
(i) $(X, d_{\mathcal{F}})$ is $\mathcal{F}$-complete,
(ii) for a point $x_0 \in X$, there exists $\alpha(x) \in (0,1]$ such that $x_1 \in [Tx_0]_{\alpha(x_0)}$ or $x_1 \in [fx_0]_{\alpha(x_0)}$ with $\beta(x_0, x_1) \geq 1$,
(iii) for all $x, y \in$ X, there exist $\psi \in \Psi$ and $\varphi \in \Phi$ such that

$$\beta(x,y)H_{\mathcal{F}}\left([Tx]_{\alpha(x)}, [fy]_{\alpha(y)}\right) \leq \psi\left(d_{\mathcal{F}}(x,y)\right) - \varphi\left(d(x,y), d\left(x, [Tx]_{\alpha_L(x)}\right), d\left(y, [Ty]_{\alpha_L(y)}\right), d\left(x, [Ty]_{\alpha_L(y)}\right), d\left(y, [Tx]_{\alpha_L(x)}\right)\right), \quad (3.23)$$

(iv) (T, f) is $\beta_{\mathcal{F}}$-admissible,
(v) for all n, $\{x_n\}$ is a sequence in X such that $\beta(x_n, x_{n+1}) \geq 1$ and $x_n \to x \in X$ as $n \to \infty$, then there exists $\beta(x_n, x) \geq 1$ for all n.
Then, T and f have a common fixed point $x^* \in [Tx^*]_{\alpha(x^*)} \cap [fx^*]_{\alpha(x^*)}$.

Proof Let L-fuzzy mappings $A, B: X \to \mathcal{F}_L(X)$ be defined as

$$Ax = X_{L_{Tx}},$$

and

$$Bx = X_{L_{fx}}.$$

Then, for all $\alpha_L(x) \in L\backslash\{0_L\}$, we get

$$[Ax]_{\alpha_L(x)} = Tx \text{ and } [Bx]_{\alpha_L(x)} = fx.$$

Hence, the remaining proof follows from Theorem 3.3.

We now consider the fixed points results for multivalued mappings.

Theorem 3.6 Let $(X, d_{\mathcal{F}})$ be $\mathcal{F}$-metric space and $\beta: (X, d_{\mathcal{F}}) \times (X, d_{\mathcal{F}}) \to [0, \infty)$. Let R be a fuzzy mapping from $(X, d_{\mathcal{F}})$ into $CB(X)$ satisfying the following:

(i) $(X, d_{\mathcal{F}})$ is $\mathcal{F}$-complete,

(ii) for all $x, y \in$ X, there exist $\psi \in \Psi$ and $\varphi \in \Phi$ such that

$$\beta(x,y)H_{\mathcal{F}}(Rx, Ry) \leq \psi\big(d_{\mathcal{F}}(x,y)\big) - \varphi\big(d(x,y), d(x,Rx), d(y,Ry), d(x,Ry), d(y,Rx)\big), \tag{3.24}$$

Then, R has a fixed point $x^* \in Rx^*$.

Proof Define L-fuzzy mappings $T: X \to \mathcal{F}_L(X)$, for some $\alpha_L \in L\backslash\{0_L\}$ by

$$T(x)(t) = \begin{cases} \alpha_L, & if\ t \in Rx, \\ 0, & if\ t \notin Rx \end{cases}$$

and

$$T(y)(t) = \begin{cases} \alpha_L, & if\ t \in Ry, \\ 0, & if\ t \notin Ry. \end{cases}$$

Then,

$$[Tx]_{\alpha_L(x)} = Rx$$

and

$$[Ty]_{\alpha_L(y)} = Ry.$$

Implies for all $x, y \in$ X,

$$H_{\mathcal{F}}\big([Tx]_{\alpha_L(x)}, [Ty]_{\alpha_L(y)}\big) = H_{\mathcal{F}}(Rx, Ry)$$

The remaining proof follows from Theorem 3.3. Thus $x^* \in X$,

$$x^* \in [Tx^*]_{\alpha_L(x^*)} = Rx^*.$$

Theorem 3.7 Let $(X, d_{\mathcal{F}})$ be $\mathcal{F}$-metric space and $\beta: (X, d_{\mathcal{F}}) \times (X, d_{\mathcal{F}}) \to [0, \infty)$. Let R_1, R_2 be fuzzy mappings from $(X, d_{\mathcal{F}})$ into $CB(X)$ satisfying the following:

(i) $(X, d_{\mathcal{F}})$ is $\mathcal{F}$-complete,

(ii) for a point $x_0 \in X$, there exists $x_1 \in R_1x_0$ or $x_1 \in R_2x_0$ with $\beta(x_0, x_1) \geq 1$,

(iii) for all $x, y \in$ X, there exist $\psi \in \Psi$ and $\varphi \in \Phi$ such that

$$\beta(x,y)H_{\mathcal{F}}(R_1x, R_2y) \leq \psi\big(d_{\mathcal{F}}(x,y)\big) - \varphi\big(d(x,y), d(x,Rx), d(y,Ry), d(x,Ry), d(y,Rx)\big), \tag{3.25}$$

(iv) (R_1, R_2) is $\beta_{\mathcal{F}}$-admissible,

(v) for all n, $\{x_n\}$ is a sequence in X such that $\beta(x_n, x_{n+1}) \geq 1$ and $x_n \to x \in X$ as $n \to \infty$, then there exists $\beta(x_n, x) \geq 1$ for all n.

Then, R_1 and R_2 have a common fixed point $x^* \in R_1x^* \cap R_2x^*$.

Proof Define L-fuzzy mappings $T, f: X \to \mathcal{F}_L(X)$, for some $\alpha_L \in L\backslash\{0_L\}$ by

$$T(x)(t) = \begin{cases} \alpha_L, & if\ t \in R_1x, \\ 0, & if\ t \notin R_1x \end{cases}$$

and

$$f(y)(t) = \begin{cases} \alpha_L, & if\ t \in R_2y, \\ 0, & if\ t \notin R_2y. \end{cases}$$

Then,

$$[Tx]_{\alpha_L(x)} = R_1x$$

and

$$[fx]_{\alpha_L(x)} = R_2y.$$

Implies for all $x, y \in$ X,

$$H_{\mathcal{F}}\big([Tx]_{\alpha_L(x)}, [fy]_{\alpha_L(y)}\big) = H_{\mathcal{F}}(R_1x, R_2y)$$

The remaining proof follows from Theorem 3.3. Thus $x^* \in X$,

$$x^* \in [Tx^*]_{\alpha_L(x^*)} \cap [fx^*]_{\alpha_L(x^*)} = R_1x^* \cap R_2x^*.$$

Application

Here, we demonstrate applicability of the results developed in the previous sections to investigate the solution of fuzzy initial value problem by using generalized Hukuhara differentiability. For more details on this, we refer to (Bede and Gal 2005, Deepmala et al. 2017, Gairola et al. 2024, Mishra et al. 2022, Seikkala 1987, Villamizar-Roa et al. 2015).

As a starting point, we introduce the symbols that will be use in this section.

Let $\mathcal{H}_c^n$ denote the space of nonempty, compact and convex subsets of the n-dimensional Euclidean space $\mathbb{R}^n$. If $A, B \in \mathcal{H}_c^n$ and $\|\cdot\|$ denotes Euclidean norm in $\mathbb{R}^n$, then, the Hausdorff metric d on $\mathcal{H}_c^n$ is define as

$$d(A,B) = max\left\{\sup_{a\in A}\inf_{b\in B}\|a-b\|, \sup_{b\in B}\inf_{a\in A}\|a-b\|\right\}.$$

and, we introduce the following definitions.

Definition 4.1 Let $u: \mathbb{R}^n \to [0,1]$ be a fuzzy mapping.

(a) u is said to be normal, if there exists $x_0 \in \mathbb{R}^n$ such that $u(x_0) = 1$.

(b) u is said to be fuzzy convex, if for all $x, y \in \mathbb{R}^n$ and $0 \le \mu \le 1$, we have

$$u(\mu x + (1-\mu)y) \ge min\{u(x), u(y)\}.$$

(c) u is said to be upper semicontinuous, if for all $\alpha \in [0,1]$, $[u]^\alpha$is closed.

(d) $[u]^0$ is compact.

Definition 4.2 (Villamizar-Roa et al. 2015) Suppose $u, v, w \in \mathcal{F}^n$. An element w is referred to as Hukuhara difference of u and v, if it satisfies the equation $u = v + w$. Now, $u \ominus_H v$ denotes the Hukuhara difference points of u and v. Clearly, $u \ominus_H u = \{0\}$, and if $u \ominus_H v$ exists, then this unique.

Definition 4.3 (Villamizar-Roa et al. 2015) Assume $g: (a,b) \to \mathcal{F}^n$ and $t_0 \in (a,b)$. g is referred to be strongly generalized differentiable or GH-differentiable at t_0, if there exists $g_G'(t_0) \in \mathcal{F}^n$ such that

$$g(t_0+h) \ominus_H g(t_0), g(t_0) \ominus_H g(t_0+h)$$

and

$$\lim_{h\to 0^+}\frac{g(t_0+h)\ominus_H g(t_0)}{h} = \lim_{h\to 0^+}\frac{g(t_0)\ominus_H g(t_0+h)}{h} = g_G'(t_0).$$

Example 4.4 (Villamizar-Roa et al. 2015) Consider the fuzzy mapping $g: \mathbb{R} \to \mathcal{F}'$ defined by $g(t) = C.t$, where C is a fuzzy number defined with its α-levels by $[C]^\alpha = [1+\alpha, 2(3-\alpha)t]$. Then,

$$[g(t)]^\alpha = \begin{cases} [1+\alpha, 2(3-\alpha)t], & t \ge 0, \\ 2(3-\alpha)t, [1+\alpha\ ], & t < 0. \end{cases}$$

Obviously, the functions g_l^α and g_r^α are not differentiable at $t = 0$. But g is GH-differentiable on $\mathbb{R}$ and $g_G'(t) = C$. That is, g is GH-differentiable at $t = 0$.

We now consider the following fuzzy initial value problem (FIVP) as follows:

$$\begin{cases} x'(t) = g\big(t, x(t)\big),\ t \in J = [0,T], \\ \qquad x(0),\ x_0, \end{cases} \tag{4.1}$$

where x' derivative is considered in the sense of GH-differentiable, where at the end points of J only one-sided derivative is considered, and the fuzzy function $g: J \times \mathcal{F}' \to \mathcal{F}'$ is continuous. The initial data x_0 in $\mathcal{F}'$. We denote $C^1(J, \mathcal{F}')$ the collections of all continuous fuzzy functions $g: J \to \mathcal{F}'$ with continuous derivative.

Lemma 4.5 (Lateef 2024) A fuzzy function $x \in C^1(J, \mathcal{F}')$ is a solution of (4.1) if and only if it verifies the integral equation

$$x(t) = x_0 \ominus_H (-1)\int_0^t g\big(s, x(s)\big)ds, \quad t \in J = [0,T].$$

(4.2)

Theorem 4.6 Suppose $g: J \times \mathcal{F}' \to \mathcal{F}'$ is continuous such that
(i) $g(t,x) < g(t,y)$, for $x < y$,
(ii) there exist some constants $\tau > 0$ large enough such that $\lambda \in \left(0, \frac{1}{2(\rho-\delta)}\right)$ and the metric for $x, y \in \mathcal{F}'$, with $x < y$ and $t \in J$ such that

$$\left\|g\big(t,x(t)\big) - g\big(t,y(t)\big)\right\|_{\mathbb{R}} \leq \tau \max_{t\in J}\left\{d_\infty(x,y)e^{-\tau(t-\delta)}\right\}$$

Then, (4.1) has a solution in $C^1(J,\mathcal{F}')$.
Proof Let $C^1(J,\mathcal{F}')$ be endowed with

$$d_\tau(x,y) = \sup_{t\in J}\max_{t\in J}\left\{d_\infty\big(x(t),y(t)\big)e^{-\tau(t-\delta)}\right\},$$

for $x, y \in C^1(J,\mathcal{F}')$ and $\tau > 0$. Then, with $g(x) = \ln(x), x > 0$ and $h = 0, (C^1(J,\mathcal{F}'), d_\tau)$ is $\mathcal{F}$ complete metric space.
Let $A, B: C^1(J,\mathcal{F}') \to (0,1]$. For $x \in C^1(J,\mathcal{F}')$,

$$L_x(t) = x_0 \ominus_H (-1)\int_0^t g\big(s,x(s)\big)ds.$$

Let $x < y$. Then, it follows from the assumption of definition 4.1 (a) that

$$L_x(t) = x_0 \ominus_H (-1)\int_\delta^t g\big(s,x(s)\big)ds < x_0 \ominus_H (-1)\int_\delta^t g\big(s,y(s)\big)ds = R_y(t)$$

If $L_x(t) \neq R_y(t)$ and $T, f: C^1(J,\mathcal{F}') \to \mathcal{F}^{C^1(J,\mathcal{F}')}$ as

$$\beta_{Tx}(r) = \begin{cases} A(x), & if\ r(t) = L_x(t) \\ 0, & otherwise. \end{cases}$$

$$\beta_{fx}(r) = \begin{cases} B(y), & if\ r(t) = L_y(t) \\ 0, & otherwise. \end{cases}$$

Again, if $\alpha_T(x) = A(x)$ and $\alpha_f(y) = B(y)$, we get

$$[Tx]_{\alpha_T(x)} = \{r \in X: (Tx)(t) \geq A(x)\} = \{L_x(t)\},$$

Similarly,

$$[fy]_{\alpha_f(y)} = \{R_y(t)\}$$

$$H\left([Tx]_{\alpha_T(x)}, [fy]_{\alpha_f(y)}\right) = max\begin{Bmatrix} x \in [Tx]_{\alpha_T(x)}, \sup\limits_{y\in[fy]_{\alpha_T(y)}} inf\|x-y\|_{\mathbb{R}} \\ y \in [fy]_{\alpha_T(y)}, \sup\limits_{x\in[Tx]_{\alpha_f(x)}} inf\|x-y\|_{\mathbb{R}} \end{Bmatrix}$$

$$\leq max\left\{\sup_{t\in J}\left\|L_x(t) - R_y(t)\right\|_{\mathbb{R}}\right\}$$

$$= \sup_{t\in J}\left\|L_x(t) - R_y(t)\right\|_{\mathbb{R}}$$

$$= \sup_{t\in J}\left\|\int_\delta^t g\big(s,x(s)\big)ds - \int_0^t g\big(s,y(s)\big)ds\right\|_{\mathbb{R}}$$

$$\leq \sup_{t\in J}\left\{\int_\delta^t \left\|g\big(s,x(s)\big) - g\big(s,y(s)\big)\right\|ds\right\}$$

$$\leq \sup_{t\in J}\left\{\int_\delta^t du\lambda\, max\left\{D_\infty(x,y)e^{-\tau(t-\delta)}\right\}ds\right\}$$

$$\leq \lambda\sup_{t\in J}\left\{(t-\delta)max\left\{D_\infty(x,y)e^{-\tau(t-\delta)}\right\}\right\}$$

$$\leq \lambda(t-\delta)d_{\mathcal{F}}(x,y)$$
$$\leq \frac{1}{2}d_{\mathcal{F}}(x,y)$$
$$= \psi\big(d_{\mathcal{F}}(x,y)\big) -$$
$$\varphi\left(d(x,y), d\big(x,[Tx]_{\alpha_L(x)}\big), d\big(y,[Ty]_{\alpha_L(y)}\big), d\big(x,[Ty]_{\alpha_L(y)}\big), d\big(y,[Tx]_{\alpha_L(x)}\big)\right).$$

Hence, all the conditions of Corollary 3.5 and Corollary 3.6 are satisfied with $\psi(t) = \frac{1}{2}t$, for $t > 0$. Thus, x^* is a solution of (4.1).

Conclusion

The main findings of this study demonstrate applicability of $\mathcal{F}$-metric space in establishing fixed point results for β-ψ- φ contractive mappings in a complete $\mathcal{F}$-metric spaces for *L*-fuzzy mappings. This study provides significant advancements in the understanding of $\mathcal{F}$-metric space, through illustrative examples, we showcased the practical applicability of the results and explored as an application, the solution for fuzzy initial-value problems. Future work could also explore the extension of this results to other types of fuzzy mappings.

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