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A Note on Paths Signatures and Convex Hulls

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Abstract

Geometrical interpretation of higher-order path signatures i.e., of order greater than two can be somewhat challenging. The purpose of this paper is to compute the volume of a convex hull using the path signature approach, specifically in the context of two-dimensional paths. We validate our findings through examples involving both cyclic and non-cyclic curves.

Keywords: Path signatures; Convex hulls; Cyclic curves; Non-cyclic curves.

Introduction

In the context of rough paths theory, the path signature is a fundamental mathematical tool that captures essential information about paths. The path signature finds applications in various domains, including data science and machine learning, where it is used to analyze time series data to interpret trends, patterns, and relationships over time. It is also utilized in human pattern recognition, disease diagnosis, and many other areas (Yang et al. 2022).

Path signatures are mathematical concepts that describe both the geometric and algebraic properties of a curve or path. Geometric properties of a path include its length, curvature, convexity, and symmetry, while algebraic properties encompass smoothness, parametric equations, and connectivity and compactness.

The concept of the path signature was originally introduced by the Chinese Mathematician Chen (1958). He defined the path signature as a sequence of iterated integrals using piecewise continuously differentiable paths. Castell and Gaines (1995) expanded on the concept of the path signature by introducing its application in solving control differential equations. Control differential equations involve external inputs and are typically represented as $(x' = f(x(t), t, u(t))$, where $u(t)$ is referred to as the control input. The primary objective of control differential equations is to enhance the performance and stability (Boedihardjo et al. 2020).

Lyons (1998) developed rough path theory and applied it to solve control differential equations with noise using the path signature. In the 2010s, researchers including Lyons and Ni (2015), Moore et al. (2019), and others made significant contributions to the advancement of the path signature's application in data science for analyzing time series data.

Arribas et al. (2018) investigated path signature in terms of application by sampling signature-based learning method to analyze the complex time series data generated by the participants' mood ratings. Results indicated that the signature methodology successfully distinguished between participant groups based on self-reported mood, outperforming standard approaches with a classification accuracy of 75%.

Hambly and Lyons (2010) refined Chen's theory and derived quantitative results applicable to paths of bounded variation. This advancement was further extended to encompass arbitrary geometric rough paths. Algorithms designed to reconstruct paths X from their signature $\sigma(X)$ have garnered significant interest due to their practical applications.

Chevyrev and Kormilitzin (2016) investigated the application of the signature method in machine learning tasks. They explored how the signature approach, as a non-parametric technique, could be utilized to extract characteristic features from data. By converting data into multi-dimensional paths using various embedding algorithms, they computed individual terms of the signature to summarize specific information contained within the data. Their research aimed to demonstrate the effectiveness of the signature method in transforming raw data into a set of features suitable for machine learning applications.

Améndola et al. (2023) explore the concept of the signature of a parametric curve, which is a sequence of tensors composed of iterated integrals. This concept is fundamental in the theory of rough paths in stochastic analysis. Their study, conducted through the perspective of algebraic geometry, introduces varieties of signature tensors for both deterministic and random paths. For deterministic paths, the focus lies on piecewise linear paths, polynomial paths, and varieties derived from free nilpotent Lie groups.

Geometrical interpretation of higher-order path signatures remains one of the core challenges in computational geometry. In this paper, we analyze in detail the technique of computing the volume of a convex hull using the path signature approach, specifically in the context of two-dimensional paths, in connection with cyclic and non-cyclic curves, and provide some examples.

Materials and Methods

This study is a pure mathematics study that primarily relies on theoretical basis. Some tools and methods used include sequences, integral operators, determinants of matrices, parametric equations of curves, set theory and mathematical induction.

Preliminaries

The following are the definitions, theorems, lemmas and propositions which are used in the establishment of main results.

Definition 1.1 (Chevyrev 2015) A path X in $\mathbb{R}^d$is a continuous mapping from some interval $[a,b]$ to $\mathbb{R}^d$

Mathematically written as $X\colon [a,b] \rightarrow \mathbb{R}^d$

For this discussion of the path signature, we will always assume that paths are piecewise continuously differentiable. This means that a path which contains derivatives of all orders.

The following is an example of piecewise continuously differentiable path.

Example 1.2. Let $X(t) = (X_1(t), X_2(t)) = (t, t^3)$ for $t \in [-2,2]$ on the left panel and $X(t) = \left(X_1\,(t), X_2\,(t)\right) = (cos\ t, sin\ t)\ for\ t \in [0,2\pi]$ on the right panel.

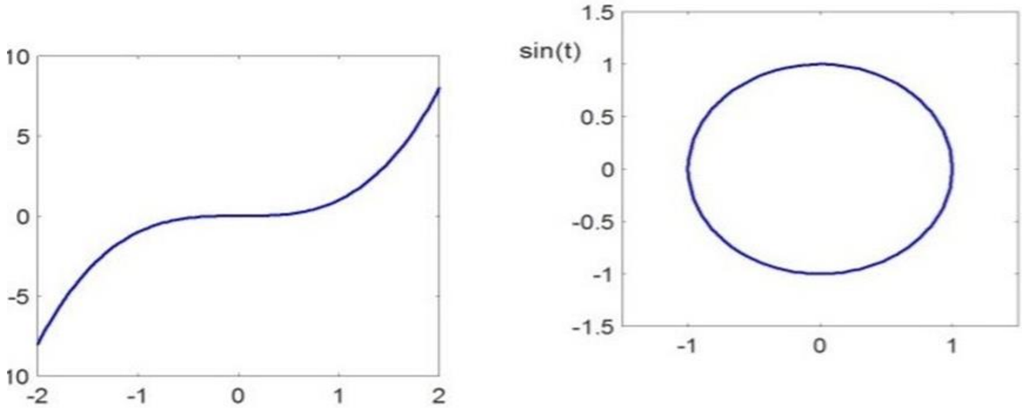


Figure 1: Example of two-dimensional piecewise continuously differentiable paths.

Definition 1.3 (Yang et al. 2022) Let $X: [0, T] \to \mathbb{R}^d$ be a path such that *X(t) = (X1(t), X2(t), X3(t), ..., Xd(t))*. For each sequence $(i_1, i_2 \ldots i_k) \in d^k$. The k-th fold iterated integrals of *X* is given by

$$S_{(i_1, i_2 \ldots i_k)} = \int_0^1 \int_0^{t_k} \ldots \int_0^{t_2} X'_{t_1}(t_i) X'_{t_2}(t_2), \ldots, X_{t_k} dt_1 dt_2 \ldots dt_k.$$

The path signature is a collection of all the iterated integrals of X.
$S(X) = (1, S_1, S_2, S_{(1,1)}, S_{(1,2)}, \ldots)$.

k is called order or level of path signature.
Let consider two-dimensional path $X(t) = (X_1(t), X_2(t))$.
$k = 1$: $S_{(1)}(X)$, $S_{(2)}(X)$,
$k = 2$: $S_{(1,1)}(X)$, $S_{(1,2)}(X)$, $S_{(2,1)}(X)$, $S_{(2,2)}(X)$,
$k = 3$: $S_{(1,1,1)}(X)$, $S_{(1,1,2)}(X)$, $S_{(1,2,1)}(X)$, $S_{(1,2,2)}(X)$, $S_{(2,1,1)}(X)$, $S_{(2,1,2)}(X)$, $S_{(2,2,1)}(X)$, $S_{(2,2,2)}(X)$.
For a 2-dimensional path the truncated signature at level 3 consists of 14 terms.

Example 2.1. (2-Dimensional Path) Consider a path $X: [0, 1] \to \mathbb{R}^2$ with $X_1(t) = t$ *and* $X_2(t) = t^2$ and the derivative $X'_1(t) = 1$ *and* $X'_2(t) = 2t$.

$$S_{(2)}(X) = \int_0^1 X'_2(t_1) dt_1 = 1,$$
$$S_{(1,2)}(X) = \int_0^1 \int_0^{t_2} X'_1(t_1) X'_2(t_2) dt_1 dt_2 = \frac{2}{3},$$
$$S_{(2,2,2)}(X) = \int_0^1 \int_0^{t_3} \int_0^{t_2} X'_1(t_1) X'_2(t_2) X'_3(t_3) dt_1 dt_2 dt_3 = \frac{1}{6}.$$

Path signature captures geometric properties of curves, although their exact interpretation for levels larger than two is not well understood.
As the dimension of a path increases, the complexity of its signature also grows significantly. While a 2-dimensional path signature consists of 14 terms, a 3-dimensional path signature will involve a much larger number of terms, making computation and geometrical interpretation more challenging (Zhou 2019).
This increase in complexity poses a substantial difficultness as we move to higher dimensions, emphasizing the need for advanced mathematical techniques and computational tools to effectively handle and analyze such signatures (Boutaib and Lyons 2022).

Definition 1.4. (Améndola et al. 2023) Let $X: [0, T] \to \mathbb{R}^2$ be a two-dimensional path such that $X(t) = (X_1(t), X_2(t))$. Alternating signature of two-dimensional path denoted as $\alpha_{(1,2)}$ is linear combination of path signatures. That is,

$$\alpha_{(1,2)} \ (X) = (S_{(1,2)} \ - S_{(2,1)} \ ).$$

Convex Hulls

Definition 1.5 (Klee 1971) The convex hull of a subset $S \subset \mathbb{R}^n$, denoted as $conv(S)$, is defined as the intersection of all convex sets in $\mathbb{R}^n$ that contain S. In other words, $conv(S)$ is the smallest convex set in $\mathbb{R}^n$ that contains all points of S.

In simple terms, imagine a set of points scattered on a plane. The convex hull is the smallest convex polygon that encompasses all of these points.

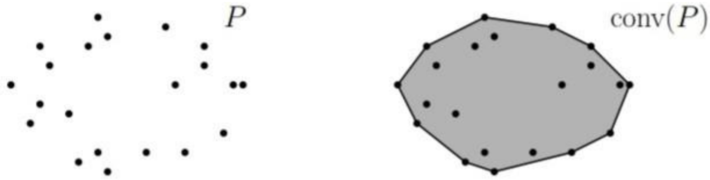


Figure 2: A point set and its convex hull.

Definition 1.6 (Berger 1990)
Convexity means that any line segment drawn between any two points on the polygon will always lie inside the polygon.

(a) A convex polygon. **(b)** A non-convex polygon.

Figure 3: Convex polygon and non-convex polygon.

Definition 1.6 (Chalopin et al. 2025) Let V denote the volume of the convex hull of a set $X \subset \mathbb{R}^d$. The volume V is defined as the measure of the smallest convex set $Conv\ (X)$ containing all points in X, where $Conv\ (X)$ is the convex hull of X in $\mathbb{R}^d$.

The volume of the convex hull, denoted $Vol\ (Conv(X))$, signifies the volume of the space.

Cyclic Curves and Non-cyclic Curves

Definition 1.6 Let $X: [0, T] \rightarrow \mathbb{R}^d$ be a path. $X(t)$ is said to be a piecewise linear path if $X(t) = (x_{k+1} - x_k)(t - k) + x_k \ \forall t \in [k, k+1]$.

Definition 1.7 (Améndola et al. 2023**)** Let $i, j, k \in \{1,2, \dots, n\}$. A curve is said to be cyclic *if* $det\ (x_j - x_i,\ x_k - x_i \geq 0$ for all $i \leq j \leq k$.

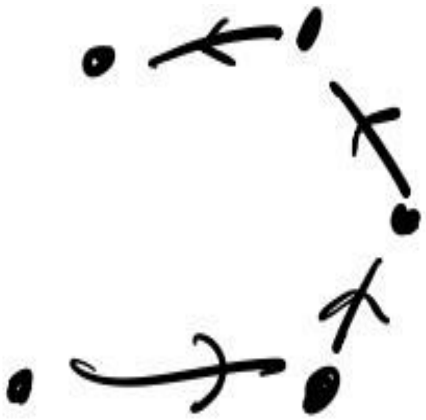
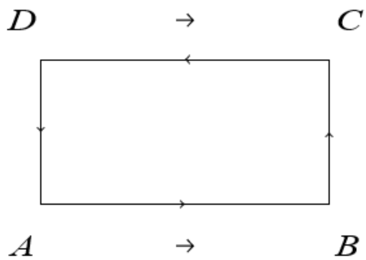


Figure 4: Example of cyclic curves.

In other words, a curve is termed cyclic if the vectors $x_1,\ x_2,\ x_3 \ldots x_n$ exhibit counter clockwise orientation.

Consider illustrative examples of cyclic curves and non-cyclic curves.
Example 1.7. Consider a rectangle $ABCD$ with the following vertices $A(0,0)$, $B(2,0), C(2,4)$ and $D(0,4)$. We need to show by using above definition it is cyclic.



Consider a path ABC. Then we formulate a matrix P by using the formula above (Definition 1.7).

$$det\ (x_j - x_i, x_k - x_i) \geq 0.$$

For ABC we have x_0, x_1 and x_2 apply above definition will be

$$det\ (x_1 - x_0, x_2 - x_0) \geq 0.$$

Given a 2×2 matrix $P = \begin{bmatrix} 2 & 2 \\ 0 & 4 \end{bmatrix}$, the determinant of P is given by:

$$det\ (P) = 8 - 0 = 8 \geq 0.$$

For BCD consider a 2×2 matrix $Q = \begin{bmatrix} 2 & 2 \\ 4 & 4 \end{bmatrix}$.
For BCD we have x_1, x_2 and x_3 apply the above definition will be

$$det\ (x_2 - x_1, x_3 - x_1) \geq 0.$$

$$Q = \begin{bmatrix} 2 & 2 \\ 4 & 4 \end{bmatrix}.$$

The determinant of the matrix is given by:

$$det(Q) = 8 - 8 = 0 \geq 0.$$

Also consider a path CDA 2×2 matrix $R = \begin{bmatrix} 2 & 0 \\ 4 & 4 \end{bmatrix}$.
For BCD, we have x_2, x_3 and x_4, then apply above definition will be

$$det\ (x_3 - x_2, x_4 - x_2) \geq 0.$$

$$R = \begin{bmatrix} 2 & 2 \\ 0 & 4 \end{bmatrix}.$$

The determinant of the matrix is given by:

$$det(R) = 8 - 0 = 8 \geq 0.$$

Therefore, for all paths ABC, BCD, and CDA, their determinants are greater than or equal to zero; hence, the rectangle ABC is cyclic.

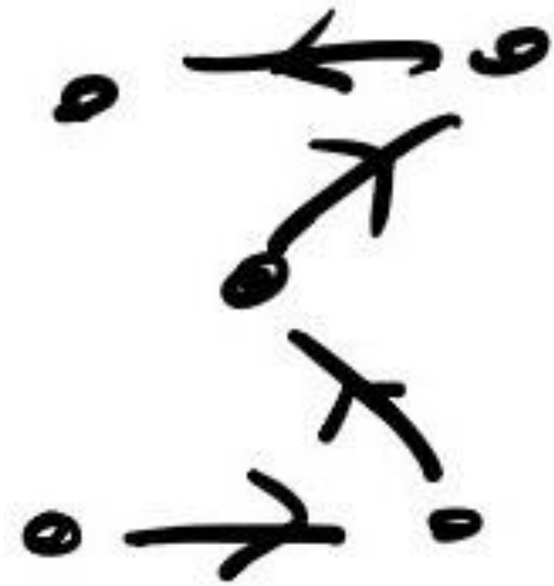


Figure 5: Example of non-cyclic curve

The relationship between path signatures, volume of convex hulls, and cyclic curves of 2-dimensional path is governed by the following theorem.

Theorem 1.9. (Améndola et al. 2023) Let X: $[0, T] \rightarrow \mathbb{R}^d$ be a cyclic curve. Then $Vol\left(Conv(x)\right) = \alpha^{(d)}(x)$.

Main Results

In this section, we present the main results of this paper

Proposition 2.1

Let X: $[0, T] \rightarrow \mathbb{R}^d$ be a linear path. The alternating signature $\alpha_{(1,2)}(X)$ is given by

$$\alpha_{(1,2)}(X) = \frac{1}{2}\sum_{k=1}^{k} \det(x_k - x_0, x_{k+1} - x_0).$$

Proof

Case 1: We consider a 2-dimensional piecewise linear path with two edges.

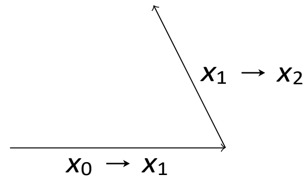


Figure 6: A diagram representing 2D piecewise linear path with two edges.

Let $v = x_1 - x_0$ and $w = x_2 - x_0$, where $v = (v_1, v_2)$ and $w = (w_1.w_2)$.
From the definition of Alternating signature:

$$\alpha_{(1,2)}(X) = \frac{1}{2}\int_0^1\int_0^{t_2}(X'_1(t_1)X'_2(t_2) - X'_2(t_1)X'_1(t_2))dt_1dt_2,$$

$$= \frac{1}{2}\int_0^1 X'_1(t_2)X'_2(t_2) - X'_2(t_2)X'_1(t_2)dt_2. \qquad (1)$$

Now, let's define a piecewise linear path $X(t)$ on $[0, 1]$

$$X(t) = \begin{cases} 2vt & if\ t \in \left[0, \frac{1}{2}\right], \\ 2w\left(t - \frac{1}{2}\right) + v & if\ t \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

Differentiate $X(t)$ to find $X'(t)$:

$$X'(t) = \begin{cases} 2v & if\ \ t \in \left[0, \frac{1}{2}\right], \\ 2w & if\ t \in [\frac{1}{2}, 1]. \end{cases}$$

$$\int_0^1 X'_1(t_2)X'_2(t_2)dt_2 = \int_0^{\frac{1}{2}} 4v_1v_2tdt + \int_{\frac{1}{2}}^1(2w_1\,(t - 1/2) + v_1)2w_2\,dt,$$

$$= \frac{1}{2}v_1v_2 + \frac{3}{2}w_1w_2 + v_1w_2. \qquad (2)$$

$$\int_0^1 X'_2(t_2)X'_2(t_1)\,dt_2 = \int_0^{\frac{1}{2}} [\![4v_2v_1tdt + \int_{\frac{1}{2}}^1(2w_2]\!]\left(t - \frac{1}{2}\right) + v_2\Big)2w_1\,dt,$$

$$= -\frac{1}{2}v_1v_2 - \frac{3}{2}w_1w_2 + v_2w_1.$$

$$\frac{1}{2}\int_0^1([\![X'_1(t_2)]\!])X'_2(t_2) - X'_2(t_2)X'_2(t_1))dt_2 = \frac{1}{2}\,v_1v_2 + \frac{3}{2}w_1w_2 + \frac{1}{2}v_1w_2 - (-\frac{1}{2}v_1v_2 + \frac{3}{2}w_1w_2 + \frac{1}{2}\,v_2w_1),$$

$$= \frac{1}{2}(v_1w_2 - v_2w_1),$$

$$= \frac{1}{2}\,\det(v, w),$$

$$= \frac{1}{2}\det(x_1 - x_0,\ x_2 - x_0).$$

Case 2: This is the second case. We consider a 2D piecewise linear path with three edges (with four points)

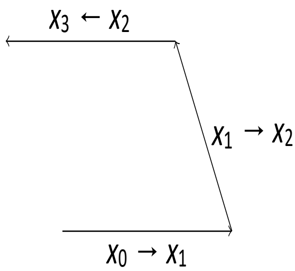


Figure 7: A diagram representing 2D piecewise linear path with three edges.

Let

$$v = x_1 - x_0, w = x_2 - x_0 \text{ and } z = x_3 - x_0.$$

Consider the definition of Alternating signature:

$$\begin{aligned} \alpha_{(1,2)} &= \frac{1}{2}\int_0^1 \int_0^{t_2} \llbracket \llbracket (X'\rrbracket_1(t_1)\rrbracket) X'_2(t_2) - X'_2 \llbracket (t)\rrbracket_1) X'_1(t_2)\big)\, dt_1 dt_2 \\ &= \frac{1}{2}\int_0^1 \llbracket X'_1(t_2)\rrbracket) X'_2(t_2) - X'_2(t_2) X'(t_2)\, dt_2 \end{aligned} \tag{1}$$

Now, let's define a piecewise linear path $X(t)$:

$$X(t) = \begin{cases} 3vt & if\ t \in [0, 1/3] \\ 3w(t - 1/3) + v & if\ t \in [1/3, 2/3] \\ 3z(t - 2/3) + w & if\ t \in [2/3, 1] \end{cases}$$

Differentiate $X(t)$ to find $X'(t)$:

$$X'(t) = \begin{cases} 3v & if\ t \in \left[0, \frac{1}{3}\right], \\ 3w & if\ t \in \left[\frac{1}{3}, \frac{2}{3}\right], \\ 3z & if\ t \in \left[\frac{2}{3}, 1\right]. \end{cases}$$

Apply the piecewise linear path to integral (1):

$$\begin{aligned} \frac{1}{2}\int_0^1 X'_1(t_2) X'_2(t_2)\, dt_2 &= \int_0^{\frac{1}{3}} 9v_1v_2\, t^2 dt + \int_{\frac{1}{3}}^{\frac{2}{3}} (2v_1w_2 + w_1w_2) dt + \int_{\frac{2}{3}}^{1} (2w_1z_2 + z_1z_2) dt, \\ &= (\frac{1}{2} v_1v_2 + \frac{1}{2}(2v_1w_2 + w_1w_2) + \frac{1}{2}(2w_1z_2 + z_1z_2) \end{aligned} \tag{2}$$

$$\frac{1}{2}\int_0^1 X'_2(t_2) X'_1(t_2) dt_2 = \frac{1}{2} v_2v_1 + \frac{1}{2}(2v_2w_1 + w_2w_1) + \frac{1}{2}(2w_1z_2 + z_1z_2) \tag{3}$$

Take equation (2) and subtract equation (3); the result will be:

$\frac{1}{2}\int_0^1 X'_1(t_2)X'_2(t_2) - X'_2(t_2)X'_1(t_2)dt_2 = \frac{1}{2}[\![(v]\!]_1 w_2 - v_2 w_1) + \frac{1}{2} w_1 z_2 - w_2 z_1.$

$$\alpha_{(1,2)}(X) = \frac{1}{2}(v_1 w_2 - v_2 w_1) + \frac{1}{2}(w_1 z_2 - w_2 z_1).$$
$$= \frac{1}{2} de\,t(v, w) + \frac{1}{2} de\,t(w, z).$$
$$= \frac{1}{2}[\det(x_1 - x_0, x_2 - x_0) + \det(x_2 - x_0, x_3 - x_0) + \det(x_3 - x_0, x_4 - x_0)].$$

By induction we can generalize for n edges.

$$\alpha_{(1,2)}(X) = \frac{1}{2}\sum_{k=1}^{k-1} \det(x_k - x_0, x_{k+1} - x_0).$$

Proof:

Base Case (*n*=2):
For 2 edges, consider a piecewise linear path with three points: x_0, x_1, x_2. The path signature $\alpha_{(1,2)}(X)$ is given by:

$$\alpha_{(1,2)}(X) = \frac{1}{2} \det(x_1 - x_0, x_2 - x_0).$$

This serves as our base case for n = 2.
Inductive Hypothesis (Assume true for $n = k$):
Let's assume that for n = k edges, the path signature can be represented as:

$$\alpha_{(1,2)}(X) = \frac{1}{2}\sum_{k=1}^{k-1} det\ (x_k - x_0, x_{k+1} - x_0).$$

Inductive Step (Prove for $n = k + 1$):
Now, we want to prove that for $n = k + 1$ edges, the path signature can be represented as:

$$\alpha_{(1,2)}(X) = \frac{1}{2}\sum_{k=1}^{k} \det(x_k - x_0, x_{k+1} - x_0).$$

Consider a piecewise linear path with $k + 1$ edges and $k + 2$ points: $x_0, x_1, x_2, \ldots, x_{k+1}$. We can break down this path into two parts: the first k edges (from x_0 to x_{k+1}) and the k-th edge to $(k + 1)$-th edge (from x_k to x_{k+1}).

By the inductive hypothesis, the path signature for the first k edges is:

$$\alpha_{(1,2)}(X) = \frac{1}{2}\sum_{k=1}^{k-1} \det(x_k - x_0, x_{k+1} - x_0).$$

Now, let's consider the k-th edge to $(k + 1)$-th edge:

$$\frac{1}{2}\int_0^1 X_1(t_{k+1})X'_2(t_{k+1}) - X_2(t_{k+1})X'_1(t_{k+1})dt_{k+1}.$$

We can expand this using the same formula we derived for 2 edges:

$$\frac{1}{2}\int_0^1 X_1(t_{k+1})X'_2(t_{k+1}) - X_2(t_{k+1})X'_1(t_{k+1})dt_{k+1} = \frac{1}{2}\det(x_k - x_0, x_{k+1} - x_0).$$

Now, by adding this to the path signature for the first k edges, we obtain the path signature for $n = k + 1$ edges:

$$\alpha_{(1,2)}(X) = \frac{1}{2}\sum_{k=1}^{k-1}\det(x_k - x_0, x_{k+1} - x_0) + \frac{1}{2}\det(x_k - x_0, x_{k+1} - x_0).$$

Simplifying the expression:

$$\alpha_{(1,2)}(X) = \frac{1}{2}\sum_{k=1}^{k}\det(x_k - x_0, x_{k+1} - x_0).$$

This completes the inductive step. Thus, we have proven that the path signature for $n = k + 1$ edges can be represented as:

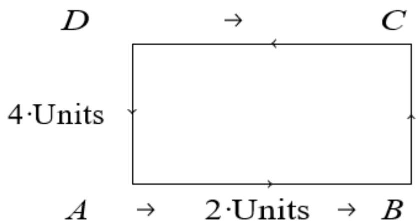
$$\alpha_{(1,2)}(X) = \frac{1}{2}\sum_{k=1}^{k}\det(x_k - x_0, x_{k+1} - x_0).$$

By induction, this holds for all positive integers n, and we have successfully generalized the path signature for n edges using sigma notation.

Verification of the results

In this subsection, we discuss the verification of the proposition mentioned above 2.1, which was obtained in this project.

Consider a rectangle ABCD with the following vertices $A(0,0)$, $B(2,0)$, $C(2,4)$ and $D(0,4)$. This rectangle is cyclic since $det\ (x_j - x_i, x_k - x_i) \geq 0$ for all $i \leq j \leq k$.



Since rectangle $ABCD$ is cyclic, we can apply the formula of path signature:

Now, let's consider the coordinates of the vertices:

$x_0 = A(0,0)$. $x_1 = B(2, 0)$. $x_2 = C(2,4)$. $x_3 = D(0,4)$.

We have four coordinate points (four vertices), so $n = 4$. The path signature of these four vertices will be computed as:

$$\alpha_{(1,2)}(X) = \frac{1}{2}\sum_{k=1}^{k-1}\det(x_k - x_0, x_{k+1} - x_0) + \frac{1}{2}\det(x_k - x_0, x_{k+1} - x_0).$$

Given a 2×2 matrix: $P = \begin{bmatrix} 2 & 2 \\ 0 & 4 \end{bmatrix}$ and $det\ (P) = 8$.

Given a 2×2 matrix: $Q = \begin{bmatrix} 2 & 2 \\ 4 & 4 \end{bmatrix}$ and $det\ (Q) = 0$.

Given a 2×2 matrix: $R = \begin{bmatrix} 2 & 0 \\ 4 & 4 \end{bmatrix}$ and $det\ (R) = 8$.

To calculate the 2D volume of the convex hull, we can equate it to the area enclosed by the rectangle ABCD which is 8 square units

Relation to Convex Hull:
Therefore, we can establish a relationship between the 2D volume of the convex hull $Vol(conv(x))$ and the area of the rectangle:

$$\alpha_{(1,2)}(x) = Vol\left(Conv(x)\right).$$

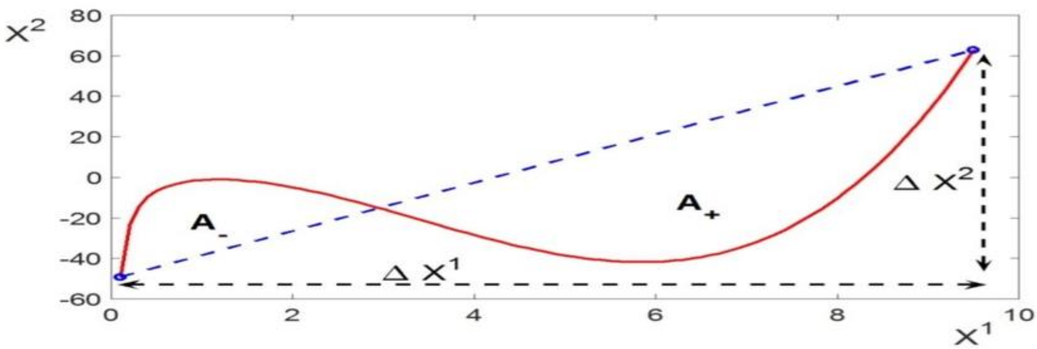
This equation illustrates that the 2D volume of the convex hull is equivalent to the area of the rectangle ABCD.

Conclusion
This section provides a comprehensive overview of the outcomes achieved through this project.

A first level signature of path is always the total displacement of path i.e. path increment. Path increment $= X_T - X_0$.
The second level of path signature contains information about the area enclosed (signed area).

If a path α is cyclic $\alpha_{(1,2)}(X) = (S_{(1,2)}(X) - S_{(2,1)}(X))$ =Area enclosed
In 2D area enclosed is a volume of convex hull.

Therefore, if a path X is cyclic the alternating signature is equal to volume of convex hull which is area in 2D i.e. $\alpha^d(x) = Vol\ (Conv(x))$.



Figue 8: Example of signed area of a curve

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Declaration
The authors declare that they have no competing interests.

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