# TIME DEPENDENT BEHAVIOR OF WAITING LINE MODELS IN DISCRETE TIME

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## Abstract

*The use of traditional closed form solutions for waiting line networks is sometimes an oversimplification, particularly where the flow of customers fluctuate with time. Two approximation methods are used to overcome this simplification. Two approximation models are developed which differ in the manner in which they model the transfer process between service nodes. Approximation I uses aggregate departure probabilities to model the transfer process. Approximation II on the other hand, takes into account the inter arrival time process to model the transfer process. Test case examples show that when the distribution of service time at node one is of negative exponential in shape, the error of the mean system size at node two for Approximation I is within 3%, whereas in approximation II is within 7%. For other types of service time distribution at node one the percentage error in the mean number of customers in system at node two is lower for approximation II than Approximation I at traffic intensities which cause frequent fluctuation on the number of customers at the first node.*

## 1. Introduction

Management of waiting lines raises a high concern in all sectors where limited resources mean that customers often use long times for service. To the customers, waiting for a long time is something that is not wanted and explains why when waiting is inevitable they join the shortest line. However, from the management point of view, short waiting lines often require resources to be under-utilized. Hence, in an effort to attempt to satisfy both parties, a balance needs to be struck such that the customers do not wait for long and the management does not incur high costs in managing the waiting lines. This whole feature of trying to balance the requirements of the customers and the management constitutes the waiting line management. Figure 1 shows a simple layout of a waiting line with one service node.

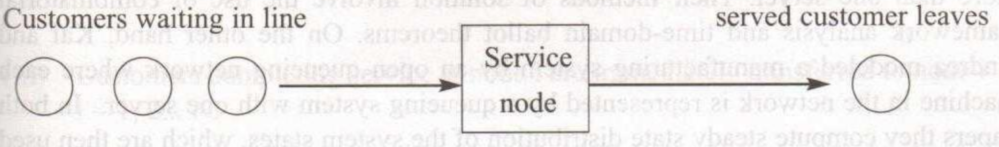


**Figure 1: Simple Waiting Line**

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Since Erlang's era to date, there has been a gradual development in the area of modeling and analysis of Waiting line systems. Such Waiting line systems fall into two broad groups namely: single stage Waiting line systems and multi stage (networks) Waiting line systems. Examples of areas where Waiting lines network systems are found include, manufacturing, communication, computers, hospitals and in offices. Waiting line networks when properly formulated can also be applied in predicting HIV/AIDS incidence in a community. Analysis of Waiting lines network systems have mainly been limited to continuous time and constant arrival process models as can be seen from (Jackson 1957, 1963), (Gordon and Newel, 1967), (Forest et al, 1975), and (Kelly, 1975, 1976). An effort to model and solve Waiting lines' networks in discrete time can be traced back from the work of (Hsu and Burke, 1976). Their work is on discrete time Waiting line model with infinite capacity buffers in tandem which can be used to model computer communication systems. Other analysts then followed. (John,1979) studied a discrete time Waiting line problem involving two queues in tandem, and unit service time, a situation that can arise within a packet switching network. (Henderson and Taylor, (1991) considered waiting lines' networks that can model 'Store and Forward' packet switching networks. Kadaba (1979, 1980) modelled a single node Waiting line system in discrete time, and extended it to a network of Waiting lines. He claims that his network model is an analogue of Jackson network but in discrete time and that it can be extended to more general networks that have no feedback, and where external arrivals occur at the first node only.

Furthermore, works on waiting lines' networks in discrete-time have been done by Jean (1983), and Gerhard and Eric, (1996). Jean's work bases on batch routing of customers through the network. Richard and Nico (1991) did a similar study to that of Jean but it is in continuous time. The product form solutions from the two papers differ. Richard and Nico explain the difference as being caused by the jobs currently in transit between stations. These jobs are not taken into account in the discrete time models, but they are taken into account in the continuous time models. In their paper Gerhard and Eric, point out that modeling of interdeparture times as a renewal process ignores their correlation. It is only valid for waiting lines' networks, which comprise M/M/1 nodes and asymptotically for traffic intensities not exceeding one.

Analytical approaches have been used by JiaFu and Khosrow (2001), Kai and Andrea, (1996) to model discrete time queueing systems. JiaFu and Khosrow have considered a discrete time queueing model with batch general arrival, deterministic service with more than one server. Their methods of solution involve the use of combinatorial framework analysis and time-domain ballot theorems. On the other hand, Kai and Andrea modeled a manufacturing system by an open queueing network where each machine in the network is represented by a queueing system with one server. In both papers they compute steady state distribution of the system states, which are then used to compute some measures of performance. Further study by (Gang and Khosrow, 2004) considered correlated queues where the service time of a packet is strongly correlated with its inter-arrival time due to the finite transmission capacity of input and output links. They used an analytical method to derive the Laplace Stieltjes Transform of the waiting time distribution and the system time distribution. Some numerical examples indicated that such correlation gives significant impact on the system performance.

Solution methods of time dependent waiting lines' networks are very sparse. Bruce and Michael (1994) studied the time dependent Waiting line network using two approximation methods. In their approach the real system is approximated in two ways. In the first case there is a principal model, which is defined as the modeler's representation that captures all but negligible features of the system, and in the second case there is an approximation model defined as a simpler version of the principal model. Bruce and Michael use a marginal decomposition approximation (MDA) to analyze the approximation model, and the correlated decomposition approximation and simulation (CDAS) method to analyze the principal model using MDA results as control variates.

Since the solution methods for discrete time waiting lines' networks that fluctuate with time are sparse, it is ideal to adopt an analytical approach to study them. The analytical approach is more appropriate during model construction stage because it enables the key parameters and their influence to be identified without the need for repeated experimentation, and also it enables determination of direction for system improvement and indicates the potential for changes.

## 2.0 Exact Model

The interest is to model a tandem Waiting line system in discrete time whereby the arrival processes is time dependent. This system is designated as an exact model. An exact model has an arrival process, which consists of individual arrivals from an infinite source to the first node only. Each node has a limited buffer to accommodate the customers, has a single server who serves the customers, one at a time, and independent of time or any other activities in the system. The assumptions used by the exact model are:

(i) The probability of states at time zero is known, for example the system starts empty with probability 1.

(ii) Customers join the system from outside at node one only.

(iii) Customers completing service at node i are immediately transferred to node i+1, i = 1, 2... N.

(iv) The arrival process of external customers to node one during the interval [t, t+1) can be described by a general distribution that may be different among intervals.

(v) There is a limit of customers allowed at each node. Excess customers are lost.

(vi) Customers must leave the system from node N after completing service.

(vii) If the server at node i is occupied, the arriving customers join the queue according to the First Come First Served (FCFS) discipline.

(viii) There is a maximum service time at each node for which a customer is served.

(ix) The service time of the customers at node i is independently identically distributed and sampled from the discrete service times, similarly at the other nodes.

Notation on exact model and its approximation models is as follows: Additional notation will be defined at the first instance of appearance. Let us consider a system consisting of two waiting lines in tandem, i.e. N = 2.

$L_i$ = The maximum number of customers including those in service at node i, i = 1, 2.

$n_i$ = Number of customers at node i, including those in service, at any epoch.

$m_i$ = The maximum service time at node i.

$y_i$ = Residual service time of the customer being served at node i.

R = Maximum number of external arrivals to node one during any interval.

r = Number of external arrivals to node one during any interval.

$v_t(r)$ = Probability of r external customers arriving at node one during the interval [t, t+1).

$s_i(j)$ = Probability that service time at node i is of length j.

$p_t(i, j, k, l)$ = Probability that the system state at time t has $n_1 = i$, $y_1 = j$, $n_2 = k$, and $y_2 = l$ where $i = 0, 1, 2, ..., L_1$; $j = 0, 1, 2, ..., m_1$; $k = 0,1,2,..., L_2$ and $l = 0, 1, 2, ..., m_2$.

## 2.1 System States

We consider a situation where the Waiting lines network consists of two nodes in tandem. In view of Brahimi, (1990) representation of the System State, Mniachi, (1997) made an extension so that it consists of four parameters and takes the form

$$\{(n_1, y_1, n_2, y_2) \text{ where } L_1 \geq n_1 \geq 0;\ m_1 \geq y_1 \geq 0;\ L_2 \geq n_2 \geq 0;\ m_2 \geq y_2 \geq 0\}$$

Associated to this system state is its occurrence probability, which we denote by

$$p_t(n_1, y_1, n_2, y_2).$$

These system states form a complete finite Markov chain. In order to determine the distribution of the System states at an epoch, a forward recurrence relation algorithm can be used. Each system state at epoch t, $(i',j',k',l')$ is considered in turn and for each possible resultant state at epoch t+1, $(i,j,k,l)$ the joint probability $p_{t+1}(i,j,k,l;\ i',j',k',l')$ is calculated as $p_{t+1}(1,j,k,l;i',j'k'l') = p_{t+1}(i,j,k,l\ /\ i',j',k',\ l')p_t(i',j',k',l')$, where $p_{t+1}(i,j,k,l;\ i',j',k',l')$ = Prob{in state $(i,j,k,l)$ at epoch t+1 and in state $(i',j',k',l')$ at epoch t} and $p_{t+1}(i,j,k,l/i',j',k',l')$ = Prob{in state $(i,j,k,l)$ at epoch t+1 given in state $(i',j',k',l')$ at epoch t}.

Once all the joint probabilities are calculated for all $(i,j,k,l)$ and all $(i',j',k',l')$, the probability distribution of the system states are obtained by simple summations of the joint probabilities as given in equation 1.

$$p_{t+1}(i,j,k,l) = \sum_{i'=0}^{L_1} \sum_{j'=0}^{m_1} \sum_{k'=0}^{L_2} \sum_{l'=0}^{m_2} p_{t+1}(i,j,k,l;i',j',k',l').$$

### *2.1.1 Steady State Results at Node One of Exact Model*

We purposely use this exact model as a research tool because it provides a standard for which results from the approximation models are compared with. It is not difficult to find known results, which can be compared with the node one results of the exact model. For example Hillier (1981) tabulated the exact results from an M/D/1 Waiting line system. To perform such a comparison a special case of Waiting line model M/D/1 was used, namely M/D/1/$L_1$. Two random arrival processes of mean arrival rates 0.1 and 0.2 and deterministic service time of unit length were used as special cases. With these values of mean arrival rates and service time, the traffic intensities at node one were 0.1 and 0.2. Table 1 shows the steady state results for the probability distribution of numbers of customers in system and the mean queue length as compared to the results from Hillier. These results show a perfect agreement.

**Table 1: Steady State Results at Node One**

(a) Probability distribution of number of customers in the system

| Number in system | Traffic intensity at node one 0.1 Exact model | 0.1 Hillier | 0.2 Exact model | 0.2 Hillier |
|---|---|---|---|---|
| 0 | 0.9000 | 0.9000 | 0.8000 | 0.8000 |
| 1 | 0.0946 | 0.0946 | 0.1771 | 0.1771 |
| 2 | 0.0051 | 0.0051 | 0.0209 | 0.0209 |
| 3 | 0.0002 | 0.0002 | 0.0018 | 0.0018 |
| 4 | 0.0000 | 0.0000 | 0.0001 | 0.0001 |
| 5 | | | 0.0000 | 0.0000 |

(b) Mean queue length

| Traffic Intensity at node one | Mean queue length Exact Model | Mean queue length Hillier |
|---|---|---|
| 0.1 | 0.0055 | 0.0055 |
| 0.2 | 0.0250 | 0.0250 |

### *2.1.2 Time Dependent Results At Node Two Of Exact Model*

For node two of the exact model, we let node one-experience random external arrivals with small mean arrival rate of 0.005 and use a general service time distribution (I), see figure 2. With this specification, node one is an $M/G/1/L_1$ Waiting line model. The resulting traffic intensity at node one is small, with a value of 0.0140. Due to the small traffic intensity at node one, the node two Waiting line system attains a steady state condition with mean arrival rate of 0.005 as well. Table 2 shows the results from node two as compared with M/G/1 results from (Alan, 1995). Comparison on the mean queue length, mean number in system and variance of queue length are made at selected epochs. Here too, the results show practically a good agreement of the two models.

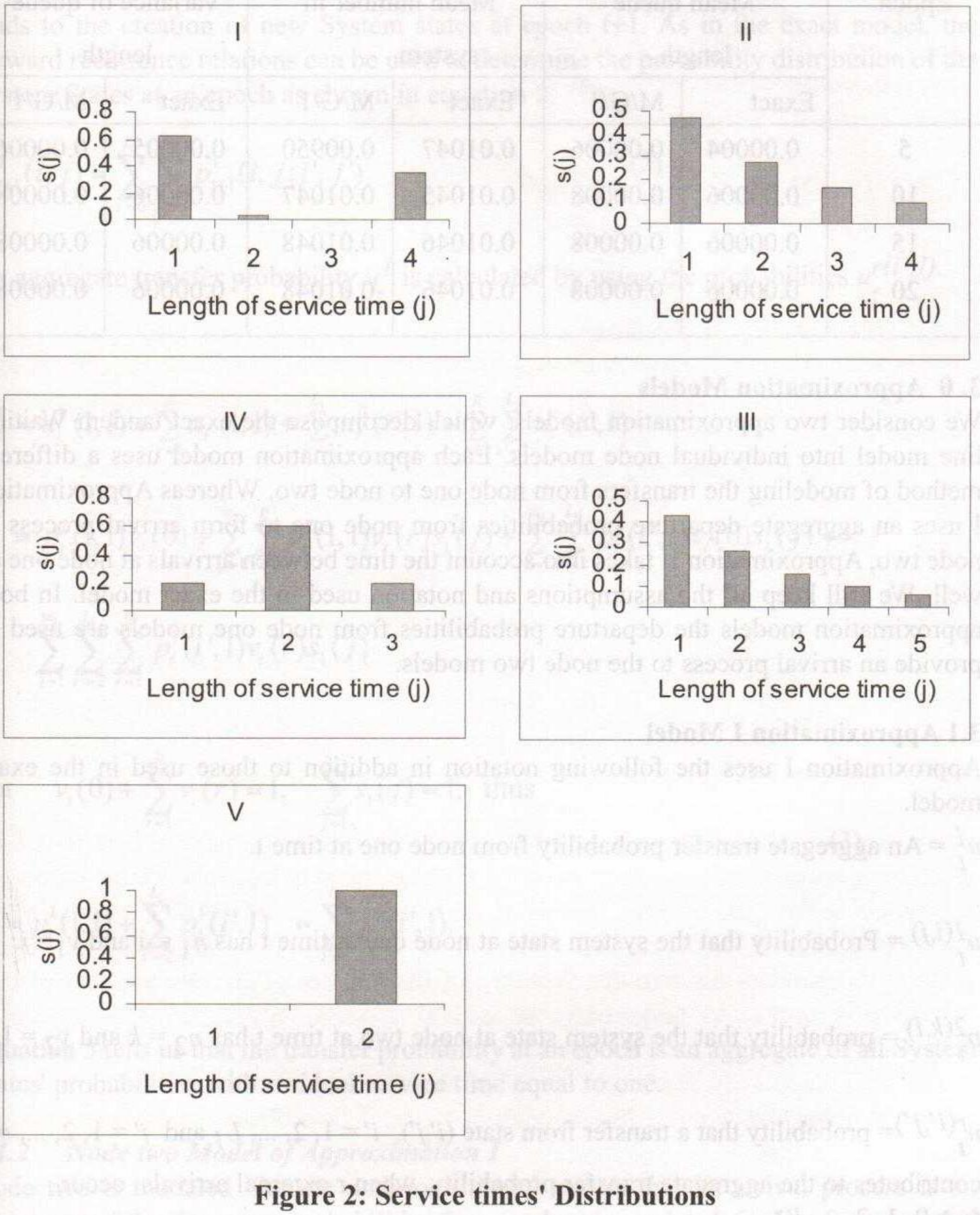


**Figure 2: Service times' Distributions**

In theory, the exact model provides a full discrete time representation of two nodes in tandem, each with a single server. In practice, there is a computational constraint because of the explosion of the System states as the capacities and service times increase. The computational draw back prompts for the development of approximation models.

**Table 2: Node two results of the exact model compared with M/G/1 results**

| Epoch | Mean queue length | | Mean number in system | | Variance of queue length | |
|---|---|---|---|---|---|---|
| | Exact | M/G/1 | Exact | M/G/1 | Exact | M/G/1 |
| 5 | 0.00004 | 0.00006 | 0.01047 | 0.00950 | 0.00005 | 0.00006 |
| 10 | 0.00006 | 0.00008 | 0.01045 | 0.01047 | 0.00006 | 0.00008 |
| 15 | 0.00006 | 0.00008 | 0.01046 | 0.01048 | 0.00006 | 0.00008 |
| 20 | 0.00006 | 0.00008 | 0.01046 | 0.01048 | 0.00006 | 0.00008 |

### 3. 0 Approximation Models

We consider two approximation models, which decompose the exact tandem Waiting line model into individual node models. Each approximation model uses a different method of modeling the transfers from node one to node two. Whereas Approximation I uses an aggregate departure probabilities from node one to form arrival process at node two, Approximation II takes into account the time between arrivals at node one as well. We still keep all the assumptions and notation used in the exact model. In both approximation models the departure probabilities from node one models are used to provide an arrival process to the node two models.

### 3.1 Approximation I Model

Approximation I uses the following notation in addition to those used in the exact model.

$u_t^1$ = An aggregate transfer probability from node one at time t.

$p_t^{1(i,j)}$ = Probability that the system state at node one at time t has $n_1$ = i and $y_1 = j$.

$p_t^{2(k,l)}$ = probability that the system state at node two at time t has $n_2 = k$ and $y_2$ = l.

$u_t^{r(i'j')}$= probability that a transfer from state $(i',j')$, $i' = 1, 2, ..., L_1$ and $j' = 1, 2, ..., m_1$ contributes to the aggregate transfer probability, when r external arrivals occur, $(r = 0, 1, 2, ..., R)$.

#### *3.1.1 Node one Model of Approximation I*

At an epoch, a vector of two parameters describe the System State and takes the form $\{(n_1,y_1)$ where $L_1 \geq n_1 \geq 0;\ m_1 \geq y_1 \geq 0\}$ with probability $p_t^1\ (n_1,y_1)$. During the time interval [t, t+1), three events can happen at node one. There can be r new arrivals with probability $v_t(r)$, new service of length j can start with probability $s_1(j)$, and there can be a reduction of service already in process with certainty. Occurrence of these events leads to the creation of new System states at epoch t+1. As in the exact model, the forward recurrence relations can be used to determine the probability distribution of the System States at an epoch as shown in equation 2.

$$p_{t+1}^1(i,j) = \sum_{i'=0}^{L_1} \sum_{j'=0}^{m_1} p_{t+1}^1(i,j;i',j') \qquad (2)$$

An aggregate transfer probability $u_t^1$ is calculated by using the probabilities $u_t^{r(i',1)}$.

$$u_t^1 = u_t^0(1,1) + \sum_{r=1}^{R} u_t^r(1,1) + \sum_{i'=2}^{L_1} u_t^0(i',1) + \sum_{r=1}^{R} \sum_{i'=2}^{L_1} u_t^r(i',1)$$

$$= p_t^1(1,1)v_t(0) + \sum_{j=1}^{m_1} \sum_{r=1}^{R} p_t^1(1,1)v_t(r)s_1(j) + \sum_{j=1}^{m_{11}} \sum_{i'=2}^{L_1} p_t^1(i',1)v_t(0)s_1(j) +$$

$$\sum_{j=1}^{m_1} \sum_{i'=2}^{L_1} \sum_{r=1}^{R} p_t^1(i',1)v_t(r)s_1(j)$$

but $\quad v_t(0) + \sum_{r=1}^{R} v_t(r) = 1, \quad \sum_{j=1}^{m_1} s_1(j) = 1;$ thus

$$u_t^1 = p_t^1(1,1) + \sum_{i'=2}^{L_1} p_t^1(i',1) = \sum_{i'=1}^{L_1} p_t^1(i',1). \qquad (3)$$

Equation 3 tells us that the transfer probability at an epoch is an aggregate of all System States' probabilities with residual service time equal to one.

### *3.1.2 Node two Model of Approximation I*

Node two is modeled like the node one model except that its arrival process is an aggregate of the departure probabilities from node one model. It is therefore modeled as a Semi-Markov chain with external input at each epoch generated by the node one model. As in node one model, a vector of two parameters' describes the System State, which takes the form $\{(n_2, y_2)$ where $L_2 \geq n_2 \geq 0;\ m_2 \geq y_2 \geq 0\}$, with probability $p_t^2(n_2, y_2)$.

At this node there are three events, which can happen during the time interval [t, t+1). A transfer can be due with probability $u_t^1$ or not due with probability 1- $u_t^1$, reduction of service in process with certainty, and start of new service with value $l$ with probability $s_2(l)$. Again, the forward recurrence relations are used to determine the probability distribution of the System States at an epoch as given by equation 4.

$$p_{t+1}^2(k,l) = \sum_{k'=0}^{L_2} \sum_{l'=0}^{m_2} p_{t+1}^2(k,l;k',l') \qquad (4)$$

### 3.2 Approximation II Model

In modeling the arrival process for Approximation I, no account was considered as to when the last arrivals occurred. In this model, we take into account the time between arrivals as well. In theory, the state of the system at an epoch would be a vector with five components. These are: the residual inter-arrival time (*riat*) to node one, the number of units at node one ($n_1$), the residual service time ($rst_1$) at node one, the number of units at node two ($n_2$) and the residual service time ($rst_2$) at node two.

With these five parameters, the representation of the System State takes the form (*riat*, $n_1$, $rst_1$, $n_2$, $rst_2$). An obvious problem with this representation is an increase in the dimension of the vector of the System State due to the parameter riat. An implementation of this formulation has not been attempted as it can be predicted that the extra dimension would cause serious computational problems.

As in Approximation I, this approximation decomposes the exact model into two nodes. Each node consists of one server working at some speed independent of the number of customers at the node or time. Service is given according to First In First Out (FIFO) queue discipline. At node one, the external arrival process is described by two known general independent discrete distributions, of the number of arrivals and that of their inter-arrival times. At node two, the arrival process is described by a discrete distribution of the time between transfers from node one. On arrival at either node, the customer immediately starts service if the server is idle. If the server is busy, the customer joins the waiting line according to the FIFO discipline if there is space otherwise it is lost.

In view of this, let us now consider a transfer process to node two, which is currently in a state of the form ($n_2$, $y_2$). If a transfer has just occurred, the probability that the next transfer will occur in the next $z$ intervals of time depends on the number of customers left at node one by the latest transfer. When the number of customers left at node one is at least one, the time to the next transfer is equal to the service time of node one, otherwise it is equal to the sum of the residual inter-arrival time plus service time.

That is

$$\text{Time until next transfer} = \begin{cases} \text{residual inter arrival time} + \text{service time at node one}, & n_1 = 0 \\ \text{service time at node one}, & n_1 \geq 1. \end{cases}$$

Approximation II is therefore making use of the distribution of the time between arrivals as well. In addition to the previous notation, Approximation II also uses the following notation.

$x_t(r)$ = conditional probability of r external arrivals to node one given that there is at least an arrival, $r = 1, 2, ..., R$.

$d_t(1+)$ = Prob(transfer occurs in [t, t+1), leaving at least one customer at node one).

$d_t(0,h)$ = Prob(transfer occurs in [t, t+1), leaving zero customers at node one, with residual inter- arrival time equal to $h$), $h = 1, 2, ..., m_1'$ where m1' is the maximum residual inter-arrival time.

$d_t$ = Probability that transfer happens in $[t, t+1)$.

$a(h)$ = Probability that the time to the next external arrival to node one is equal to $h$.

$b(z)$ = Probability that a new inter-transfer time to node two is equal to $z$, $z = 1, 2, ..., m_1' + m_1$.

$p_t^1(h,n_1,y_1)$ = Probability that the System State at node one at time $t$ has $h$ remaining time to the next external arrival, $n_1$ customers and $y_1$ remaining service time.

$p_t^2(z,n_2,y_2)$ = Probability that the System State at node two at time $t$ has $z$ remaining time to the next transfer, $n_2$ customers and $y_2$ remaining service time.

### *3.2.1 Node One Model Of Approximation II*

A three-parameter vector represents the System State of node one model and takes the form $\{h,n_1,y_1,)$ where $m_1 \geq h \geq 1,\ L_1 \geq n_1 \geq 0;\ m_1 \geq y_1 \geq 0\}$ which, happens with probability $p_t^1(h,n_1,y_1)$.

At this node, four events can take place in the time interval [t,t+1). There can be $r$ external arrivals, which occur with probability $x_t(r)$, start of new inter-arrival time with value h with probability a($h$), reduction of residual service time of service in progress which occur with certainty and start of new service with value $j$ with probability $s_1(j)$. The process of determining the probability distribution of the System States at an epoch is achieved by using the forward recurrence relations as in previous models, which is given by equation 5.

$$p^1_{t+1}(h,i,j) = \sum_{h'=1}^{m_1'} \sum_{i'=0}^{L_1} \sum_{j'=0}^{m} p'_{t+1}(h,i,j;h',i',j') \qquad (5)$$

The Probability that a transfer occurs in the time interval [t, t+1), leaving zero customers at node one, with residual inter-arrival time equal to $h'$-1 is computed by using equation 6.

$$d_t(0,h'-1) = p^1_t(h',1,1), \qquad m_1' > h' > 1. \qquad (6)$$

On the other hand, the Probability that a transfer occurs in the time interval [t, t+1), leaving at least one customer at node one is computed by using equation 7.

$$d_t(1+) = \sum_{i'=1}^{L_1}\sum_{r=1}^{R}\sum_{h=1}^{m_1'}\sum_{j=1}^{m_1} p^1_t(1,i',1)x_t(r)a(h)s_1(j) + \sum_{h'=2}^{m_1}\sum_{i'=2}^{L_1}\sum_{j=1}^{m_1} p^1_t(h',i',1)s_1(j)$$
$$= \sum_{i'=1}^{L_1} p^1_t(1,i',1) + \sum_{h'=2}^{m_1'}\sum_{i'=2}^{L_1} p^1_t(h',i',1). \qquad (7)$$

Since these two events are independent, then, the ultimate probability that a transfer happens during the time interval [t, t+1) is the sum of probabilities obtained from equations 6 and 7, which is

$$d_t = d_t(0,h'-1) + d_t(1+) \qquad (8)$$

With this transfer probability, in equation 8 we can then work out the probability distribution of the inter-transfer time random variable, which, the node two model uses. This is given by equation 9

$$b(z) = \begin{cases} \dfrac{s_1(z)d_t(1+) + \sum_{h'=2}^{z-1} d_t(0,h'-1)s_1(z-h'+1)}{d_t}, & \text{if } n_1 = 0 \\ s_1(z), & \text{if } n_1 \geq 1. \end{cases} \qquad (9)$$

### *3.2.2 Node two model of Approximation II*

System State representation of node two model is similar to that of node one model and takes the form $\{(z,n_2,y_2)$ where $m'_1+m_1 \geq z \geq 1, L_2 \geq n_2 \geq 0; m_2 \geq y_2 \geq 0\}$, happening with probability $p_t^2(z,n_2,y_2)$. Change of System states at this node is similar to that of node one model. Four events can take place in the time interval [t, t+1).

A transfer from node one is due when the inter-transfer time drops to zero. Start of new inter-transfer time with value Z with probability $b(z)$, start of new service time with value $l$ with probability $s_2(l)$ and, a reduction of residual service time of service in progress, which, occur with certainty. The process of determining the probability distribution of the System States is achieved by applying the forward recurrence relations as given by equation 10. All forward recurrence relationships mentioned in this paper have been implemented in C++ programming language. In the next section, we provide numerical examples, which compare the approximation results for selected measures of performance.

$$p_{t+1}^{2}(z,k,l)=\sum_{z'=1}^{m_1'+m_1}\sum_{k'=0}^{L_{21}}\sum_{l'=0}^{m_2}p_{t+1}^{2}(z,k,j;z',k',l') \tag{10}$$

## 4.0 Test Case Examples

In order to compare the node one results of the approximation models with those from the exact model, a set of discrete time arrival processes to node one were used together with a typical unimodal II service time. The resulting traffic intensities ranged from 0.4 to 1.0. At the 95 epoch, irrespective of the formulation procedures all the models, the mean system size and variance of queue length at node one agree perfectly as shown in figure 3.

In the case of node two models' results, some discrete time arrival processes were used whereby the arrivals were allowed to happen after every five units of time with certainty. The arrival distributions were generated using Alan's (1995) software. In order to use this software, you only need to know the mean and variance of the arrival process. At node one, three different discrete service time distributions (III, IV, V) were used. These service times represent different types of discrete distributions. The distribution III is a typical example, which is close to negative exponential family, IV represents a unimodal and symmetric distribution that can be common in real situations, and V represents a constant distribution. The effects of these service times at node two as used at node one, were investigated by keeping the same service time at node two, III in this case.

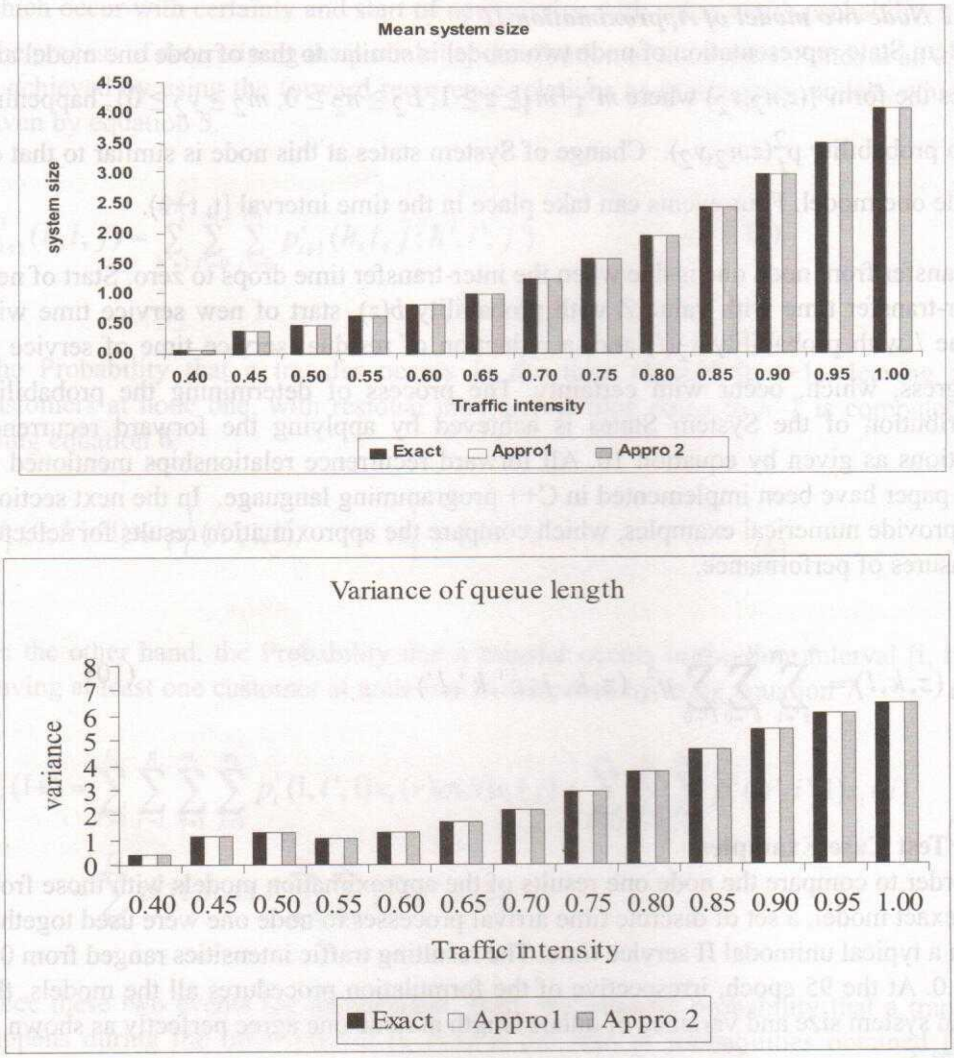


**Figure 3: Node One Results**

Figure 4 shows the bar charts of the percentage errors on the mean system size and variance of queue length results from the node two models of Approximations II and I, relative to the exact model results. In figure 4(a) and (b), service time distribution III is used at both nodes one and two, (c) and (d) uses service time distribution IV at node one and III at node two, and in (e) and (f) service time distribution V is used at node one and III at node two. These results show that the performance of Approximation I is good when node one uses a service time which has a negative exponential shape and for the medium to high traffic intensities, see figure 4(a) and (b). For example the percentage error of the mean number in system and variance of queue length for medium and high traffic intensities are within six percent. It is also observed that Approximation I performs better than Approximation II at medium to high traffic intensities irrespective of the type of service time used at node one model; for example at 0.70, 0.75, 0.95 and 1.0 traffic intensities. Of particular note is that when a unimodal and symmetric service time (IV) is used at node one, Approximation I under-approximates the mean system sizes at node two. These results also show that a change of service time at node one from negative exponential (memoryless) results into big error (at low traffic intensity) in the mean system size at node two of Approximation I model, see figure 4(c) and (e).

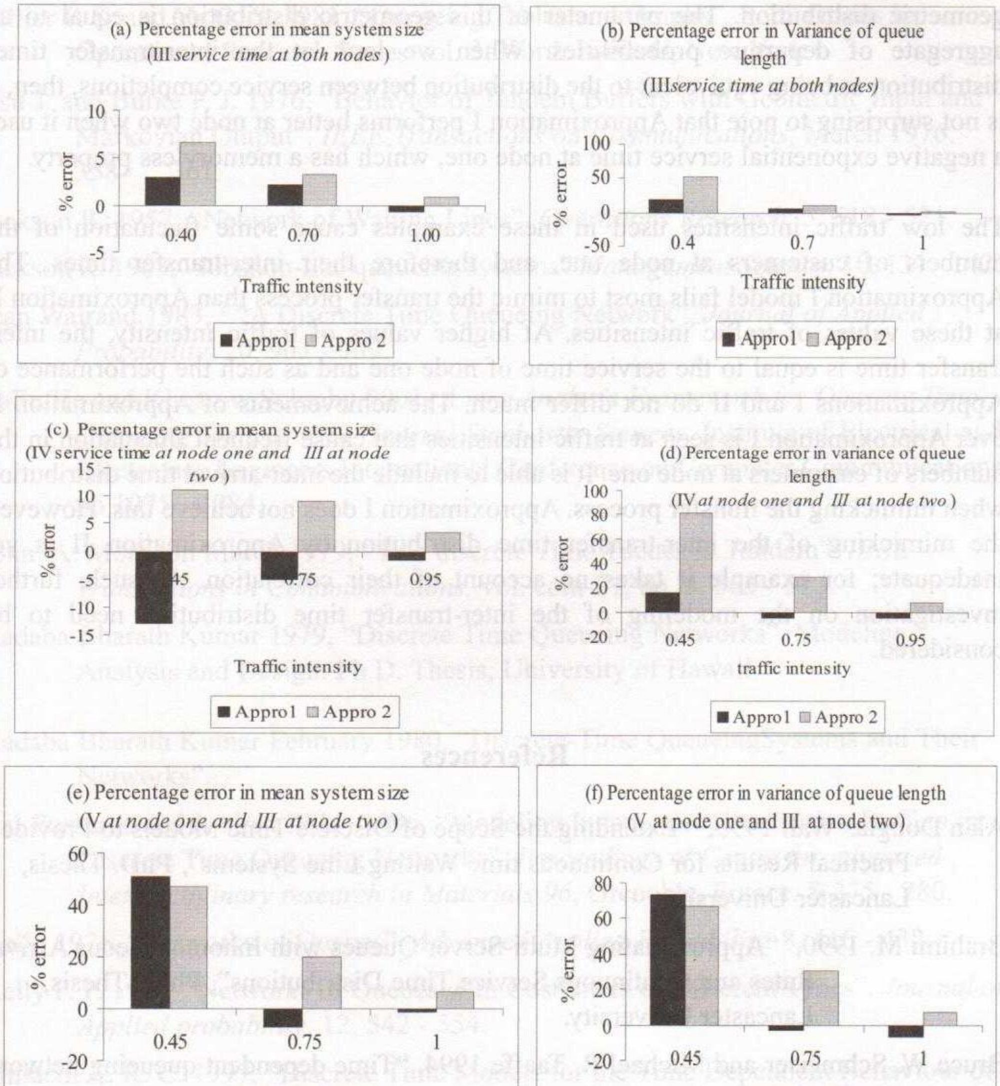


**Figure 4: Node Two Results**

**5.0 Conclusion**

All test cases' examples on the performance of Approximations I and II have shown one thing in common. When the traffic intensity at node one is high, they perform consistently better. The marked difference between the two approximation models is at lower traffic intensities. The behaviors of Approximation I can be explained by looking at the way the inter-transfer times to node two are modeled. At any epoch, node two can have a transfer to or no transfer. The transfer process is modeled by following a Bernoulli distribution and that the time between transfers is seen as following a geometric distribution. The parameter of this geometric distribution is equal to an aggregate of departure probabilities. When we look at the inter-transfer times distribution as being equivalent to the distribution between service completions, then, it is not surprising to note that Approximation I performs better at node two when it uses a negative exponential service time at node one, which has a memoryless property.

The low traffic intensities used in these examples cause some fluctuation of the numbers of customers at node one, and therefore their inter-transfer times. The Approximation I model fails most to mimic the transfer process than Approximation II at these values of traffic intensities. At higher values of traffic intensity, the inter-transfer time is equal to the service time of node one and as such the performance of Approximations I and II do not differ much. The achievements of Approximation II over Approximation I is seen at traffic intensities that cause frequent fluctuation in the numbers of customers at node one. It is able to include the inter-arrival time distribution when mimicking the transfer process. Approximation I does not achieve this. However, the mimicking of the inter-transfer time distribution by Approximation II is yet inadequate; for example it takes no account of their correlation. As such, further investigation on the modeling of the inter-transfer time distribution need to be considered.

## References

Alan Douglas Wall 1995, "Extending the Scope of Discrete Time Models to Provide Practical Results for Continuous time Waiting Line Systems", PhD. Thesis, Lancaster University.

Brahimi M. 1990, "Approximating Multi-Server Queues with Inhomogeneous Arrival Rates and Continuous Service Time Distributions", Ph.D. Thesis, Lancaster University.

Bruce W. Schmeiser and Michael R. Taaffe 1994, "Time dependent queueing network approximations as simulation external control variates", *Operations Research letters* 16, 1 - 9.

Gang Uk Hwang and Khosrow Sohraby 2004, "Performance of Correlated Queues: the Impact of Correlated Service and Inter-arrival Times". *Performance Evaluation,* Volume 55, Issues 1-2, Pages 129-145.

Gerhard Hasslinger and Eric S. Rieger 1996, "Analysis of Open Discrete-Time Queueing Networks. A Refined Decomposition Approach". *Journal of Operations Research Society* volume 47 no 5, 640 - 653.

Gordon W. J. and Newell G. F. 1967, "Closed Queueing Systems With Exponential Servers", *Operations Research,* 15, 254 - 265.

Henderson W. Taylor P. G. 1991, "Some New Results on Queueing Networks With Batch Movement", *Journal of Applied Probability* 28, 409 - 421.

Hillier F. S. and Yu O. S. 1981, "Queueing Tables and Graphs". Publications in Operational Research Series vol. 3, North Holland, New York.

Hsu J. and Burke P. J. 1976, "Behavior of Tandem Buffers with Geometric Input and Markovian Output". *IEEE Transactions on Communications,* March 1976, 358- 361.

Jackson R. 1957, "Network of Waiting Lines". *Operations Research,* 5: 518 - 521.

Jackson R. 1963, "Jobshop-like queueing systems" *Management Science,* 10, 131 - 142.

Jean Walrand 1983, : "A Discrete Time Queueing Network", *Journal of Applied Probability.* 20, 903 - 909.

JiaFu He and Khosrow Sohraby 2001, *A new Analysis Framework for Discrete Time Queueing Systems with General Stochastic Sources.* Institute of Electrical and Electronics Engineers International Conference on Computer Communications pp 1075 - 1084.

John A. Morrison March 1979, "Two discrete Time Tueues in Tandem", *IEEE Transactions in Communications,* vol. com-27, no. 3, 563 - 573.

Kadaba Bharath Kumar 1979, "Discrete Time Queueing Networks", Modeling, Analysis and Design. Ph.D. Thesis, University of Hawaii.

Kadaba Bharath Kumar February 1980, "Discrete-Time QueueingSystems and Their Networks".

Kai Furmans and Andrea Zillus 1996, "Modeling Independent Production Buffers in Discrete Time Queueing Networks". *Proceedings of Centre for advanced Interdisciplinary research in Materials 96*, Grenoble, France. S 275 - 280.

Kelly 1976, "Networks of Queues". *Advanced Applied Probability.* 8, 416 - 432.

Kelly F. P. 1975, "Networks of Queues with Customers of Different Types". *Journal of Applied probability.* 12, 542 - 554.

Mniachi A. R. C. 1997, "Discrete Time Models for the Time Dependent Behaviour of Queues in Series". Ph D. Thesis, Lancaster University.

Richard J. Boucherie and Nico M. Van Dijk 1991, "Product Forms for Fueueing Networks with State Dependent Multiple Job Transitions". *Advanced Applied Probability.* 23, 152 - 187.